Complexity dichotomies for list homomorphism problems of signed graphs

Jan Bok [1], Richard C. Brewster [2], Tomás Feder, Pavol Hell [3], Nikola Jedličková [1], and Arash Rafiey [4]

[1] Charles University, Prague, Czech Republic
[2] Thompson Rivers University, Kamloops, BC, Canada
[3] Simon Fraser University, Burnaby, BC, Canada
[4] Indiana State University, Terre Haute, IN, USA

24th July 2024

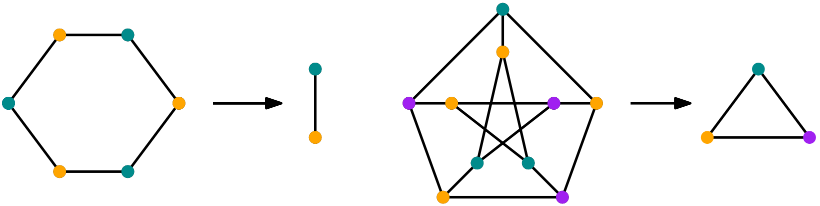
ISMP 2024, session Complexity and algorithmic aspects of structured families of graphs

Graph homomorphism

Definition

A **graph homomorphism** between two graphs G and H is a mapping $f : V(G) \to V(H)$ such that for every edge $uv \in E(G)$, $f(u)f(v) \in E(H)$.

- Graph homomorphisms are **adjacency-preserving mappings** between the vertex sets of two graphs.

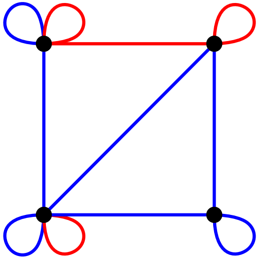


Signed graphs

Definition (Signed graph)

A **signed graph** is a graph G (with possible loops and multiple edges) together with a mapping $\sigma : E(G) \to \{+, -\}$, assigning a sign to each edge of G.

Moreover, at most two loops per vertex and at most two edges between a pair of vertices are allowed and different loops (resp. multiple edges) with the same endpoints have different signs.



We often refer to positive edges as blue edges, negative edges as red edges, and a pair of positive and negative edge as bicoloured edge.

Switching operation

Definition (Switching and switching equivalence)

The **switching operation** can be applied to any vertex of a signed graph. It results in multiplying signs of all its incident edges by -1.

Two graphs are **switching-equivalent** if we can obtain one from the other by a sequence of switchings.

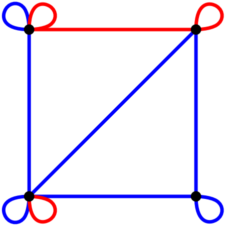


Figure: An example of switching-equivalent graphs.

Definition (Switching and switching equivalence)

The **switching operation** can be applied to any vertex of a signed graph. It results in multiplying signs of all its incident edges by -1.

Two graphs are **switching-equivalent** if we can obtain one from the other by a sequence of switchings.

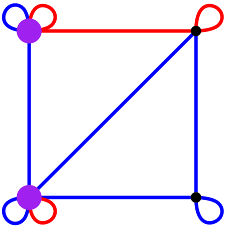
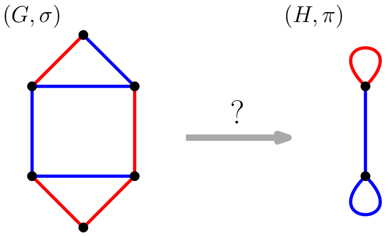


Figure: An example of switching-equivalent graphs.

Definition

We say that a mapping $f : V(G) \to V(H)$ is a **homomorphism** of the signed graph (G, σ) to the signed graph (H, π), if there exists a signed graph (G, σ'), switching equivalent to (G, σ), such that

- whenever uv is a positive edge in (G, σ'), then (H, π) contains a positive edge joining $f(u)$ and $f(v)$, and
- whenever uv is a negative edge in (G, σ'), then (H, π) contains a negative edge joining $f(u)$ and $f(v)$.

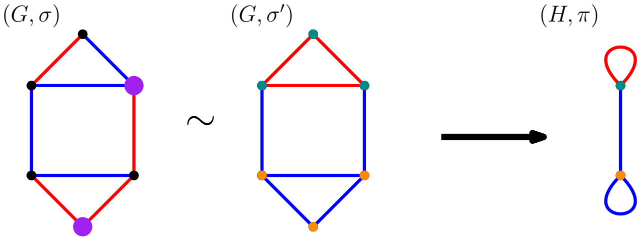


Homomorphism of signed graphs

Definition

We say that a mapping $f : V(G) \to V(H)$ is a **homomorphism** of the signed graph (G, σ) to the signed graph (H, π), if there exists a signed graph (G, σ'), switching equivalent to (G, σ), such that

- whenever uv is a positive edge in (G, σ'), then (H, π) contains a positive edge joining $f(u)$ and $f(v)$, and
- whenever uv is a negative edge in (G, σ'), then (H, π) contains a negative edge joining $f(u)$ and $f(v)$.



Signed core and dichotomy for S-HOM((H, π))

Problem

Let (H, π) be a fixed signed graph. The problem S-HOM((H, π)) is defined as follows:

INPUT: A signed graph (G, σ).
QUESTION: Is there a homomorphism of (G, σ) to (H, π)?

Problem

Let (H,π) be a fixed signed graph. The problem $\text{S-Hom}((H,\pi))$ is defined as follows:

INPUT: A signed graph (G,σ).
QUESTION: Is there a homomorphism of (G,σ) to (H,π)?

The dichotomy of $\text{S-Hom}((H,\pi))$ was settled by Brewster and Siggers (previously conjectured by Brewster, Foucaud, Hell, and Naserasr).

Definition (Signed core)

A signed graph (G,σ) is a **signed core** (or **s-core**) if every homomorphism $f : (G,\sigma) \to (G,\sigma)$ is a bijection.

Signed core and dichotomy for $\textsc{S-Hom}((H,\pi))$

Problem

Let (H,π) be a fixed signed graph. The problem $\textsc{S-Hom}((H,\pi))$ is defined as follows:

INPUT:	A signed graph (G,σ).
QUESTION:	Is there a homomorphism of (G,σ) to (H,π)?

The dichotomy of $\textsc{S-Hom}((H,\pi))$ was settled by Brewster and Siggers (previously conjectured by Brewster, Foucaud, Hell, and Naserasr).

Definition (Signed core)

A signed graph (G,σ) is a **signed core** (or **s-core**) if every homomorphism $f:(G,\sigma)\to(G,\sigma)$ is a bijection.

Theorem (Brewster, Siggers, 2018)

The problem $\textsc{S-Hom}((H,\pi))$ *is polynomial if* (H,π) *has an s-core with at most 2 edges and NP-complete otherwise.*

Problem

Let (H, π) be a fixed signed graph. The LIST-S-HOM((H, π)) problem is defined as follows:

INPUT:	A signed graph (G, σ) with lists $L(v) \subseteq V(H)$ for every $v \in V(G)$.
QUESTION:	Does there exist a homomorphism f from (G, σ) to (H, π) such that $f(v) \in L(v)$ for every $v \in V(G)$?

List variant of the problem

Problem

Let (H, π) be a fixed signed graph. The LIST-S-HOM$((H, \pi))$ problem is defined as follows:

INPUT:	A signed graph (G, σ) with lists $L(v) \subseteq V(H)$ for every $v \in V(G)$.
QUESTION:	Does there exist a homomorphism f from (G, σ) to (H, π) such that $f(v) \in L(v)$ for every $v \in V(G)$?

- It is clear that every NP-complete case of S-HOM$((H, \pi))$ remains NP-complete for LIST-S-HOM$((H, \pi))$.
- We can and will focus on signed graphs (H, π) whose s-cores have at most two edges.
- **Warning: Occasional lies and simplifications ahead.**

Our goals

The dichotomy conjecture for CSP was proved by Bulatov and Zhuk. Our problem **can be formulated as a CSP** and therefore, we know that there is a dichotomy in terms of complexity.

Our goals

The dichotomy conjecture for CSP was proved by Bulatov and Zhuk. Our problem **can be formulated as a CSP** and therefore, we know that there is a dichotomy in terms of complexity.

However, here we are interested in graph-theoretical classifications instead of algebraic ones.

The dichotomy conjecture for CSP was proved by Bulatov and Zhuk. Our problem **can be formulated as a CSP** and therefore, we know that there is a dichotomy in terms of complexity.

However, here we are interested in graph-theoretical classifications instead of algebraic ones.

So far, we had results on **trees** and on **separable graphs**. We have dichotomies in terms of forbidden induced subgraphs + structural descriptions of polynomial target graphs.

Here I will focus on complexity dichotomy for list homomorphism problem in the case of reflexive and irreflexive weakly balanced signed graphs.

Weakly balanced signed graphs

Definition (Negative and positive cycles)

The sign of a cycle is the product of the signs of its edges. We distinguish between a **negative** cycle and a **positive** cycle.

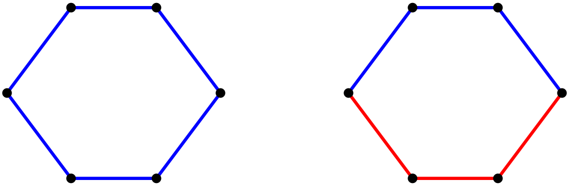


Figure: A negative cycle on the left and a positive cycle on the right.

Definition (Balanced and anti-balanced graph)

We say that a signed graph is **balanced** if every cycle in the graph is positive and **anti-balanced** if each cycle has an even number of positive edges.

Definition (Balanced and anti-balanced graph)

We say that a signed graph is **balanced** if every cycle in the graph is positive and **anti-balanced** if each cycle has an even number of positive edges.

- A signed graph is balanced if and only if it is switching-equivalent to a signed graph with all edges positive. (Analogously for anti-balanced graphs.)

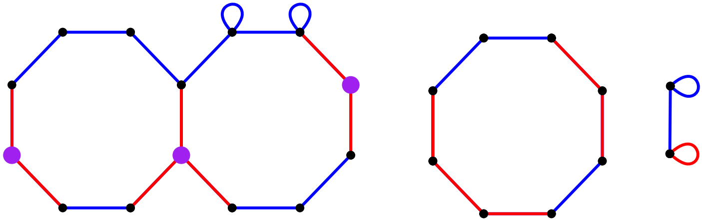


Figure: A balanced graph on the left. Two non-balanced graphs on the right.

Weak balancedness

Definition (Weakly balanced and anti-balanced graph)

We say that a signed graph is **weakly balanced** (**weakly anti-balanced**) if it is switching-equivalent to a graph in which all edges are bicoloured or blue (respectively red).

Definition (Weakly balanced and anti-balanced graph)

We say that a signed graph is **weakly balanced** (**weakly anti-balanced**) if it is switching-equivalent to a graph in which all edges are bicoloured or blue (respectively red).

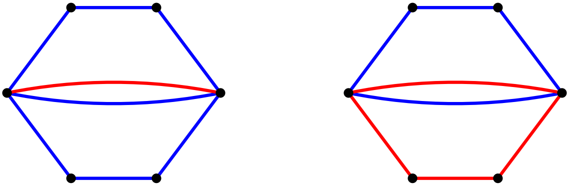


Figure: A weakly balanced (and also weakly anti-balanced) graph on the left. A weakly unbalanced graph on the right.

The main result

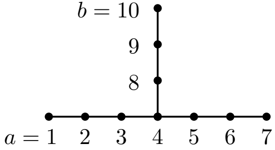
Theorem

- *A weakly balanced bipartite signed graph $\widehat{H}$ has a special min ordering if and only if it has no chain and no invertible pair.*
- *Furthermore, if $\widehat{H}$ has a special min ordering, then the list homomorphism problem for $\widehat{H}$ can be solved in polynomial time. Otherwise $\widehat{H}$ has a chain or an invertible pair and the list homomorphism problem for $\widehat{H}$ is NP-complete.*

Ingredient 1: invertible pairs

Definition

An **invertible pair** in $\widehat{H}$ is a pair of vertices a, b, with two walks of the same length — U with vertices $a = u_0, u_1, \ldots, u_k = b$ and D with vertices $b = d_0, d_1, \ldots, d_k = a$, such that for each i, $0 \leq i \leq k-1$, both $u_i u_{i+1}$ and $d_i d_{i+1}$ are edges of $\widehat{H}$, while $d_i u_{i+1}$ is not an edge of $\widehat{H}$.



$$U = 1 - 2 - 3 - 4 - 5 - 6 - 7 - 6 - 7 - 6 - 5 - 4 - 8 - 9 - 10$$

$$D = 10 - 9 - 10 - 9 - 10 - 9 - 8 - 4 - 3 - 2 - 1 - 2 - 1 - 2 - 1$$

Theorem

If (H, π) has an invertible pair, then $\textsc{List-S-Hom}((H, \pi))$ *is NP-complete.*

Definition

Let (U, D) be two walks in $\widehat{H}$ of equal length, say k, U with vertices $u = u_0, u_1, \ldots, u_k = v$ and D with vertices $u = d_0, d_1, \ldots, d_k = v$. We say that (U, D) is a **chain**, provided $uu_1, d_{k-1}v$ are unicoloured edges and $ud_1, u_{k-1}v$ are bicoloured edges, and for each i, $1 \leq i \leq k-2$, we have

1. both $u_i u_{i+1}$ and $d_i d_{i+1}$ are edges of $\widehat{H}$ while $d_i u_{i+1}$ is not an edge of $\widehat{H}$, or
2. both $u_i u_{i+1}$ and $d_i d_{i+1}$ are bicoloured edges of $\widehat{H}$ while $d_i u_{i+1}$ is not a bicoloured edge of $\widehat{H}$.

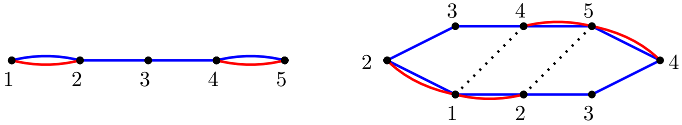


Figure: The graph (H, π) (left) and a chain in it (right).

Theorem

If a signed graph $\widehat{H}$ contains a chain, then LIST-S-HOM($\widehat{H}$) *is NP-complete.*

Ingredient 3: special bipartite min orderings

Definition

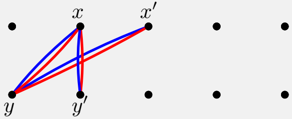
A **special bipartite min ordering** of the bipartite graph $\widehat{H}$ with parts B, W is a pair of linear orderings $<_b$ of B and $<_w$ of W, such that

Ingredient 3: special bipartite min orderings

Definition

A **special bipartite min ordering** of the bipartite graph $\widehat{H}$ with parts B, W is a pair of linear orderings $<_b$ of B and $<_w$ of W, such that

- for any vertices $x <_w x'$ in W and any two vertices $y <_b y'$ in B, if $xy', x'y$ are both edges in $\widehat{H}$, then xy is also an edge in $\widehat{H}$ (**underbar property**), and

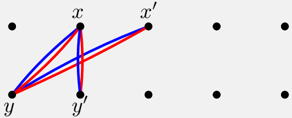


Ingredient 3: special bipartite min orderings

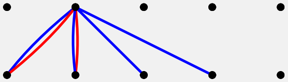
Definition

A **special bipartite min ordering** of the bipartite graph $\widehat{H}$ with parts B, W is a pair of linear orderings $<_b$ of B and $<_w$ of W, such that

- for any vertices $x <_w x'$ in W and any two vertices $y <_b y'$ in B, if $xy', x'y$ are both edges in $\widehat{H}$, then xy is also an edge in $\widehat{H}$ (**underbar property**), and



- the bicoloured neighbours of any vertex appear before its unicoloured neighbours in both orderings (**special property**).



Sketch of the proof

Theorem (Part I)

A weakly balanced bipartite signed graph $\widehat{H}$ has a special min ordering if and only if it has no chain and no invertible pair.

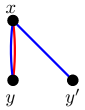
- If H contains an invertible pair, there is no min ordering. If $\widehat{H}$ has a chain, speciality cannot be achieved.
- Suppose from now on, that our graph has no invertible pairs and no chains.

Sketch of the proof

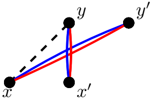
Theorem (Part I)

A weakly balanced bipartite signed graph $\widehat{H}$ has a special min ordering if and only if it has no chain and no invertible pair.

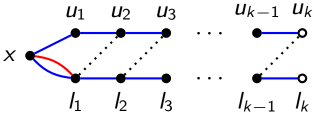
- If H contains an invertible pair, there is no min ordering. If $\widehat{H}$ has a chain, speciality cannot be achieved.
- Suppose from now on, that our graph has no invertible pairs and no chains.
- We find a set D of pre-ordered pairs with respect to **special** and **underbar** property:



$xy, xy' \Rightarrow y < y'$



$x < x' \Rightarrow y' < y$

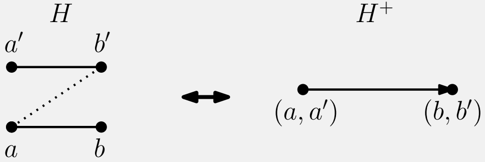


$$l_1 < u_1 \Rightarrow l_2 < u_2 \Rightarrow \ldots \Rightarrow l_k < u_k$$

Definition

The **pair digraph** H^+ for unsigned bipartite graphs H, with a fixed bipartition A, B:

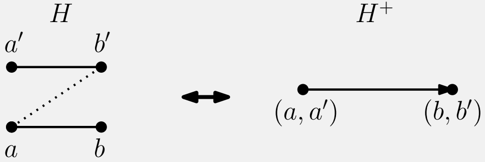
- $V(H^+) = \{(a, a') : a, a' \in A, a \neq a'\} \cup \{(b, b') : b, b' \in B, b \neq b'\}$
- $E(H^+)$: (a, a') to (b, b') if and only if $ab, a'b'$ are edges of H while ab' is not an edge of H.



Definition

The **pair digraph** H^+ for unsigned bipartite graphs H, with a fixed bipartition A, B:

- $V(H^+) = \{(a, a') : a, a' \in A, a \neq a'\} \cup \{(b, b') : b, b' \in B, b \neq b'\}$
- $E(H^+) : (a, a')$ to (b, b') if and only if $ab, a'b'$ are edges of H while ab' is not an edge of H.



Extension theorem

Suppose D is a set of ordered pairs of distinct vertices of a bipartite graph H that is closed under **reachability and transitivity**. Then there exists a bipartite min ordering $<$ of H such that $x < y$ for each $(x, y) \in D$ if and only if H has no invertible pair.

Extension theorem + no chains

Extension theorem restated

Suppose D is a set of ordered pairs of distinct vertices of a bipartite graph H that is closed under reachability and transitivity. Then there exists a bipartite min ordering $<$ of H such that $x < y$ for each $(x, y) \in D$ if and only if H has no invertible pair.

Theorem

If $\widehat{H}$ has no chain, then the set D can be extended to a special bipartite min ordering.

Extension theorem + no chains

Extension theorem restated

Suppose D is a set of ordered pairs of distinct vertices of a bipartite graph H that is closed under reachability and transitivity. Then there exists a bipartite min ordering $<$ of H such that $x < y$ for each $(x, y) \in D$ if and only if H has no invertible pair.

Theorem

If $\widehat{H}$ has no chain, then the set D can be extended to a special bipartite min ordering.

Both theorems together imply that **without invertible pairs and chains, there is a special bipartite min ordering**.

Sketch of the proof

Theorem (Part II)

If a weakly balanced bipartite signed graph $\widehat{H}$ has a special min ordering, then the list homomorphism problem for $\widehat{H}$ can be solved in polynomial time. Otherwise $\widehat{H}$ has a chain or an invertible pair and the list homomorphism problem for $\widehat{H}$ is NP-complete.

Sketch of the proof

Theorem (Part II)

If a weakly balanced bipartite signed graph $\widehat{H}$ has a special min ordering, then the list homomorphism problem for $\widehat{H}$ can be solved in polynomial time. Otherwise $\widehat{H}$ has a chain or an invertible pair and the list homomorphism problem for $\widehat{H}$ is NP-complete.

- Let us have a special bipartite min ordering $\leq$ for $\widehat{H}$.
- Our algorithm finds a homomorphism of the underlying graphs and if there is a negative cycle mapped to a positive closed walk, then **we can modify lists of** $\widehat{G}$ (thanks to a technical lemma) and search for a better one.

Sketch of the proof

Theorem (Part II)

If a weakly balanced bipartite signed graph $\widehat{H}$ has a special min ordering, then the list homomorphism problem for $\widehat{H}$ can be solved in polynomial time. Otherwise $\widehat{H}$ has a chain or an invertible pair and the list homomorphism problem for $\widehat{H}$ is NP-complete.

- Let us have a special bipartite min ordering $\leq$ for $\widehat{H}$.
- Our algorithm finds a homomorphism of the underlying graphs and if there is a negative cycle mapped to a positive closed walk, then **we can modify lists of** $\widehat{G}$ (thanks to a technical lemma) and search for a better one.

Technical lemma

Let $\widehat{H}$ be a weakly balanced bipartite signed graph with a special bipartite min ordering $\leq$. Let C be a closed walk in $\widehat{G}$ and f, f' are two homomorphisms of $\widehat{G}$ to $\widehat{H}$ such that

- $f(v) \leq f'(v)$ for all vertices v of $\widehat{G}$, and
- $f(C)$ contains only blue edges but $f'(C)$ contains a bicoloured edge.

Then the homomorphic images $f(C)$ and $f'(C)$ are disjoint.

Conclusion and outlook

- trees (B., Brewster, Feder, Hell, Jedličková $\rightarrow$ MFCS 2020 + Discrete Mathematics)
- separable (B., Brewster, Feder, Hell, Jedličková, 2021 $\rightarrow$ CALDAM 2021 + Theoretical Computer Science)

Conclusion and outlook

- trees (B., Brewster, Feder, Hell, Jedličková $\rightarrow$ MFCS 2020 + Discrete Mathematics)
- separable (B., Brewster, Feder, Hell, Jedličková, 2021 $\rightarrow$ CALDAM 2021 + Theoretical Computer Science)
- weakly balanced reflexive signed graphs (Kim, Siggers, 2021) — *complementary results to ours*
- **weakly balanced irreflexive + reflexive signed graphs** (B., Brewster, Hell, Jedličková, Rafiey $\rightarrow$ LATIN 2022 + Algorithmica 2024).

Conclusion and outlook

- trees (B., Brewster, Feder, Hell, Jedličková $\rightarrow$ MFCS 2020 + Discrete Mathematics)
- separable (B., Brewster, Feder, Hell, Jedličková, 2021 $\rightarrow$ CALDAM 2021 + Theoretical Computer Science)
- weakly balanced reflexive signed graphs (Kim, Siggers, 2021) — *complementary results to ours*
- **weakly balanced irreflexive + reflexive signed graphs** (B., Brewster, Hell, Jedličková, Rafiey $\rightarrow$ LATIN 2022 + Algorithmica 2024).
- **general irreflexive signed graphs** — *current work in progress, new type of obstructions (trichains)*

- trees (B., Brewster, Feder, Hell, Jedličková → MFCS 2020 + Discrete Mathematics)
- separable (B., Brewster, Feder, Hell, Jedličková, 2021 → CALDAM 2021 + Theoretical Computer Science)
- weakly balanced reflexive signed graphs (Kim, Siggers, 2021) — *complementary results to ours*
- **weakly balanced irreflexive + reflexive signed graphs** (B., Brewster, Hell, Jedličková, Rafiey → LATIN 2022 + Algorithmica 2024).
- **general irreflexive signed graphs** — *current work in progress, new type of obstructions (trichains)*

We are now close to a general characterisation. It seems that polynomial cases are rare beyond weakly balanced.

- trees (B., Brewster, Feder, Hell, Jedličková → MFCS 2020 + Discrete Mathematics)
- separable (B., Brewster, Feder, Hell, Jedličková, 2021 → CALDAM 2021 + Theoretical Computer Science)
- weakly balanced reflexive signed graphs (Kim, Siggers, 2021) — *complementary results to ours*
- **weakly balanced irreflexive + reflexive signed graphs** (B., Brewster, Hell, Jedličková, Rafiey → LATIN 2022 + Algorithmica 2024).
- **general irreflexive signed graphs** — *current work in progress, new type of obstructions (trichains)*

We are now close to a general characterisation. It seems that polynomial cases are rare beyond weakly balanced.

Thank you for attention!

`https://janbok.github.io`

Funded by the European Union (ERC, POCOCOP, 101071674). Views and opinions expressed are however those of the author(s) only and do not necessarily reflect those of the European Union or the European Research Council Executive Agency. Neither the European Union nor the granting authority can be held responsible for them.

Funded by the European Union

European Research Council
Established by the European Commission