Spectrahedral shadows and completely positive maps on real closed fields

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Joint work with Mario Kummer and Andreas Thom

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Outline

1. Semialgebraic feasibility problems
2. Spectrahedral shadows
3. Conjecture of Helton-Nie, solved by Scheiderer
4. New model theoretic proof, new results
5. Open problems

Linear Program Feasability

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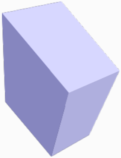
Satisfiability of Linear Inequalities:



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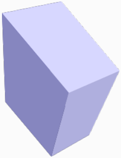

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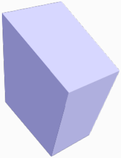

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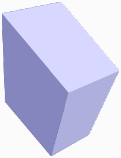

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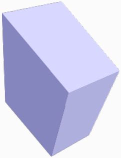

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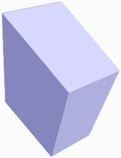

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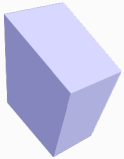

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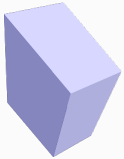

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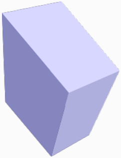

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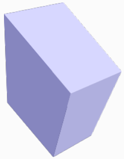

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What other forms of constraints can be solved in polynomial time as well?

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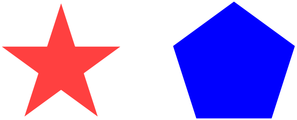
NP-hard (no polynomial-time algorithm unless P=NP).

Convexity



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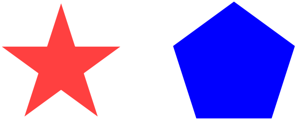

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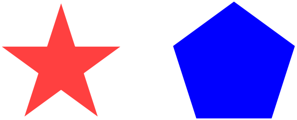

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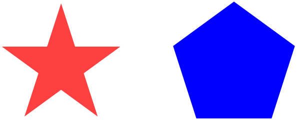

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Example of semialgebraic convex set:

$$\{(x,y) \in \mathbb{R}^2 \mid y \geq x^2\}$$

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Example. All constraints of the form

$$x + y = z$$
$$x \leq y$$
$$x = 1$$

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A primitive positive (pp) formula is a formula of the form

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Question. Is there a set of 'basic' convex semialgebraic sets that can pp-define all others?

Spectrahedra

$A \in \mathbb{R}^{k \times k}$ real symmetric matrix.

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This talk: model-theoretic approach to these questions.

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$\mathfrak{A}$: a structure. Let $\mathfrak{A}^*$ be an elementary extension of $\mathfrak{A}$,
i.e., $\mathfrak{A}$ is a substructure of $\mathfrak{A}^*$ such that for every first-order formula $\phi(x_1,\ldots,x_k)$ and $a \in A^k$ we have $\mathfrak{A} \models \phi(a)$ if and only if $\mathfrak{A}^* \models \phi(a)$.

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So: f positive definite iff $T_f(b^2) > 0$ for all $b \in R[\Lambda] \setminus \{0\}$.

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Then $T = T_f$ and f is positive definite. □

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Question. Is there $p \in \mathbb{R}[x_1, \dots, x_n]$ such that

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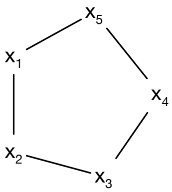

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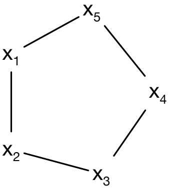

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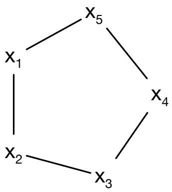

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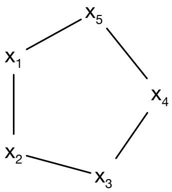

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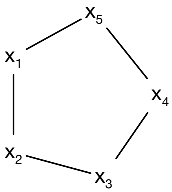

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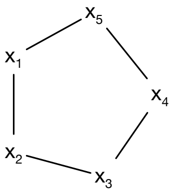

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Theorem implies: $P_+(S)$ not spectrahedral shadow. □

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9. $L_f(q)(\epsilon_1^{-1}, \ldots, \epsilon_n^{-1}) = T_f(p) < 0$.
10. Thus, L_f preserves all relations of $\mathfrak{B}$,
    but not the formula defining $P_+(S)$. □

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