A Complexity Dichotomy in Spatial Reasoning via Ramsey Theory

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May 2024

European Research Council
Established by the European Commission

ERC Synergy Grant POCOCOP (GA 101071674).

Overview

- Spatial Reasoning Formalism RCC5
- Complexity Dichotomy
- Tools:
  1. Universal algebra (Polymorphisms)
  2. Model theory (Homogeneous Structures)
  3. Ramsey theory (Extreme Amenability)
  4. Finite-domain CSP dichotomy of Bulatov and Zhuk.
- Outlook

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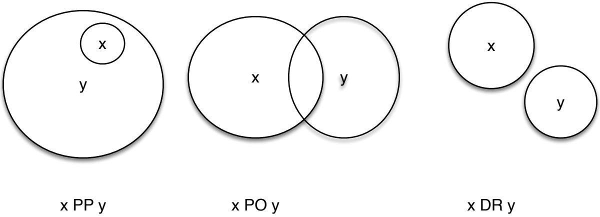

- One of the fundamental formalisms for spatial reasoning
- RCC stands for region connection calculus.
- Formally, a relation algebra with 5 atoms.
- Idea: variables denote non-empty regions in space.
- 5 binary relations between regions:
  - *x* PP *y*: *x* is a proper part of *y*.
  - *x* DR *y*: *x* is disjoint region to *y*.
  - *x* PO *y*: *x* properly overlaps with *y*.

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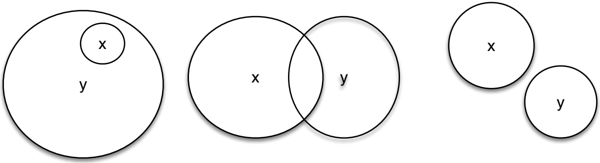

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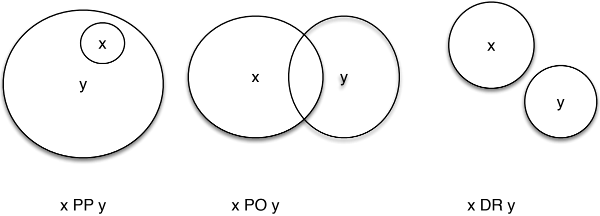

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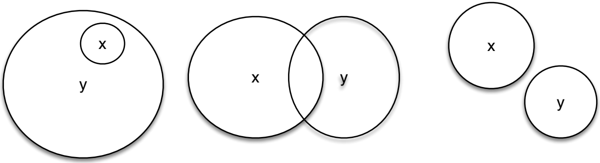

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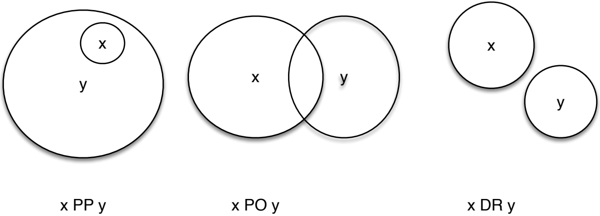

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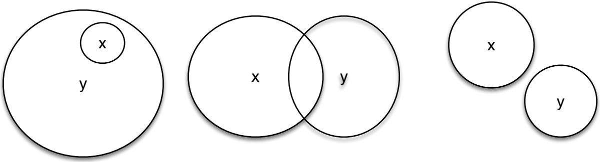

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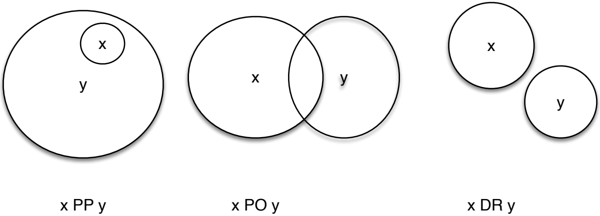

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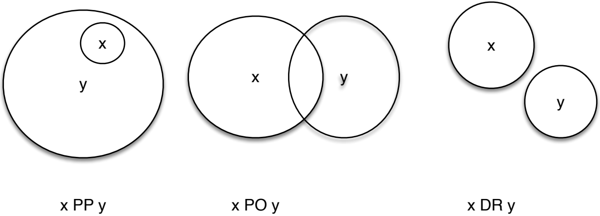

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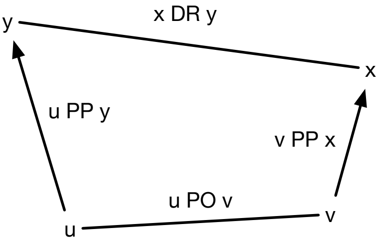

Input: Finite set of variables,
conjunction of constraints of the form *x* PP *y*, *x* DR *y*, or *x* PO *y*.

Task: Decide whether there are non-empty regions that satisfy all the constraints.

Question: What is the computational complexity of this problem?

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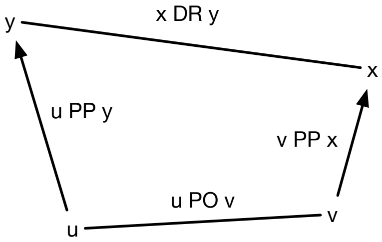

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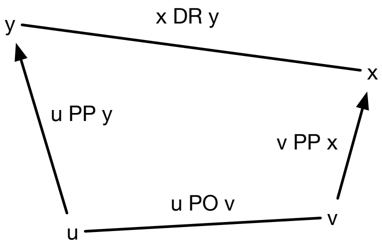

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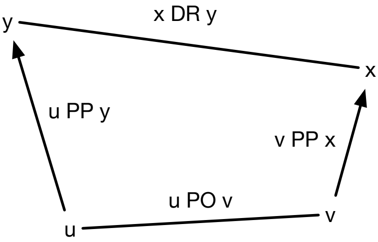

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Let $\mathfrak{S}$ be the following relational structure:

- The domain is the set S of all non-empty subsets of $\mathbb{N}$.
- The signature is $\{\text{PP}, \text{PO}, \text{DR}\}$
- $\text{PP}^{\mathfrak{S}} := \{(x, y) \mid x \subset y\}$.
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The constraint satisfaction problem (CSP) for $\mathfrak{S}$:
Input: a finite conjunction ϕ of atomic $\{\text{PP}, \text{PR}, \text{PO}\}$-formulas.
Question: Is ϕ satisfiable in $\mathfrak{S}$?

For $R \subseteq S^2$, write $R^{\smile}$ for $\{(y, x) \mid (x, y) \in R\}$.

Note. $\underbrace{\{\text{PP}^{\mathfrak{S}}, (\text{PP}^{\mathfrak{S}})^{\smile}, \text{DR}^{\mathfrak{S}}, \text{PO}^{\mathfrak{S}}, =\}}_{\text{5 basic relations}}$ partition S^2.

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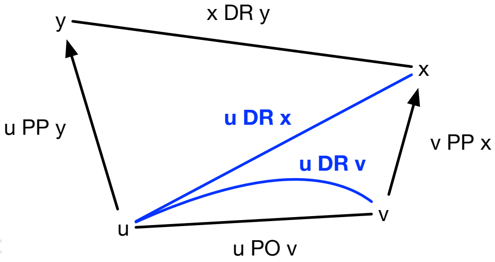

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- Can be phrased using Datalog:

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- Datalog more general than PC: allow auxiliary predicates of arbitrary arity

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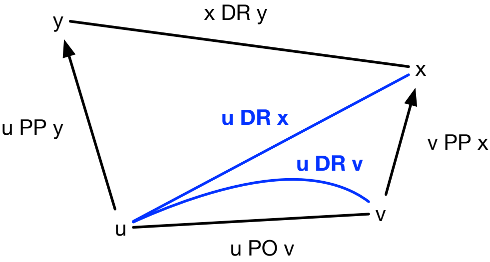

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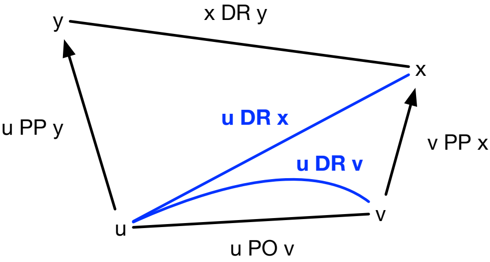

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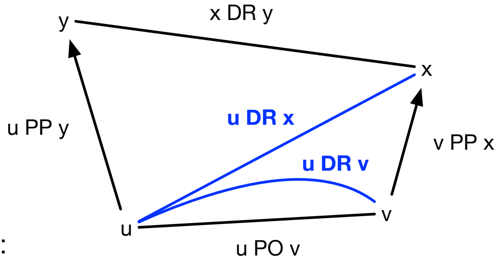

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$$\text{goal} \quad :- \quad x_1 \text{ DR' } x_2, \; x_1 \text{ PO } x_2$$

$$\dots$$

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Other Spatial Reasoning Problems

- Network satisfaction problem for RCC5: additionally allow constraints of the form $\bigvee_{R \in \mathcal{R}} x \; R \; y$ for $\mathcal{R} \subseteq \{\text{PP}, \text{PP}^{\smile}, \text{PO}, \text{DR}, =\}$.
- Can be modelled as CSP($\mathfrak{T}$) where $\mathfrak{T}$ is an expansion of $\mathfrak{S}$ by all unions of the 5 basic relations.
- Nebel+Renz'97: CSP($\mathfrak{T}$) is NP-complete.
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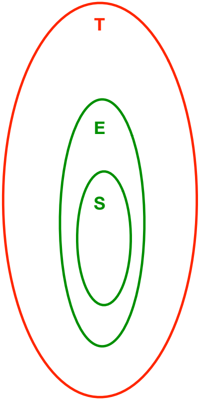

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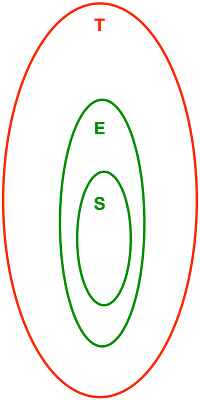

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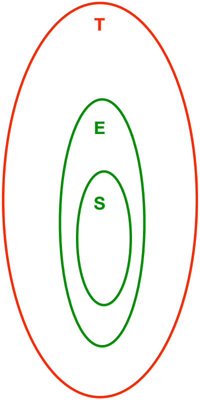

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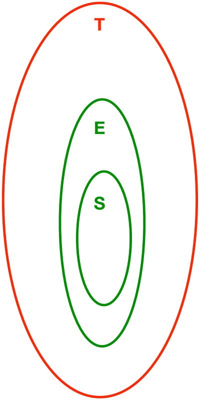

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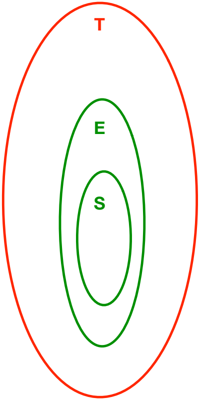

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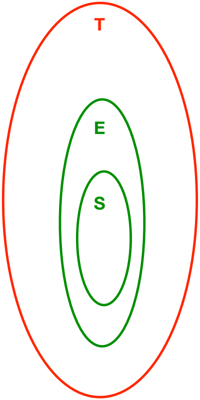

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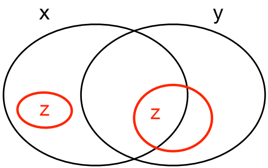
$\mathfrak{B}$: expansion of $\mathfrak{S}$ by relations R with a quantifier-free definition in $\mathfrak{S}$.



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- in P?
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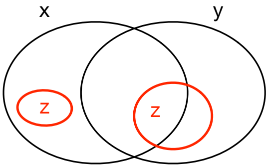

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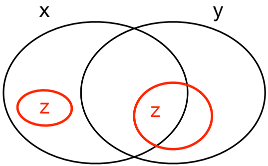

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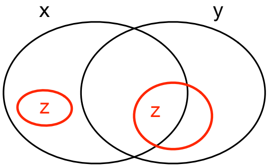

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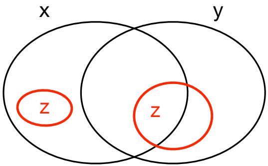

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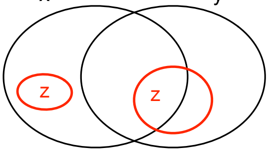

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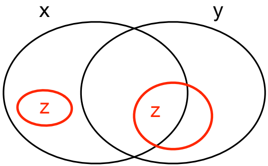

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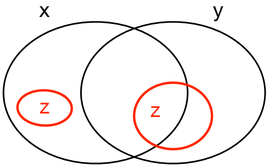

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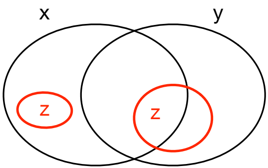

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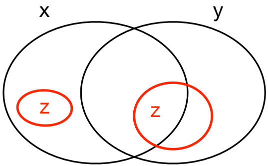

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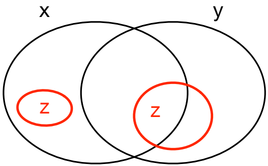

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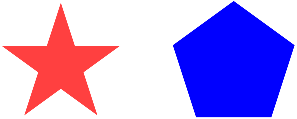
Polymorphisms generalise automorphisms.



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$(f(a^1_1, \dots, a^k_1), \dots, f(a^1_m, \dots, a^k_m)) \in R$.

Examples:

- $(x, y) \mapsto (x + y)/2$ preserves all convex relations $R \subseteq \mathbb{R}^m$.
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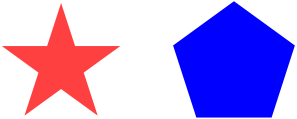

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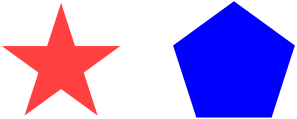

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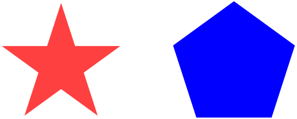

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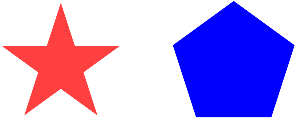

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Tool 2: Finite-Domain Dichotomy

Let $\mathfrak{B}$ be a structure with a finite domain B.

Theorem (Bulatov'17, Zhuk'17/20).

If $\mathfrak{B}$ has a polymoprhism $f \colon B^k \to B$ which is cyclic, i.e., $k \geq 2$ and for all $x_1, \ldots, x_k \in B$

$$f(x_1, \ldots, x_k) = f(x_2, \ldots, x_k, x_1)$$

then CSP($\mathfrak{B}$) is in P.
Otherwise, CSP($\mathfrak{B}$) is NP-hard.

Examples.

- $(x, y) \mapsto \max(x, y)$ is a polymorphism of $(\{1, \ldots, n\}; <)$.
  Hence, CSP$(\{1, \ldots, n\}; <)$ can be solved in polynomial time.
- For for $n \geq 3$, every polymorphism of $(\{1, \ldots, n\}; \neq)$ is of the form $(x_1, \ldots, x_k) \mapsto g(x_i)$ for some $i \in \{1, \ldots, k\}$ and some $g \in S_n$.
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Theorem (Bulatov'17, Zhuk'17/20).

If $\mathfrak{B}$ has a polymoprhism $f \colon B^k \to B$ which is cyclic, i.e., $k \geq 2$ and for all $x_1, \ldots, x_k \in B$

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Infinite Domains

How about $\mathfrak{B}$ with infinite domains?

- The polymorphisms of $\mathfrak{B}$ determine the complexity of CSP($\mathfrak{B}$) if $\mathfrak{B}$ is ω-categorical (B+Nešetřil'03):
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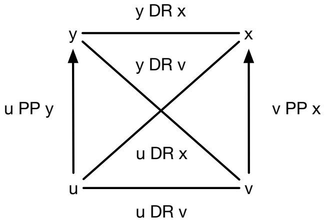

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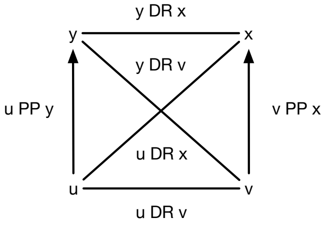

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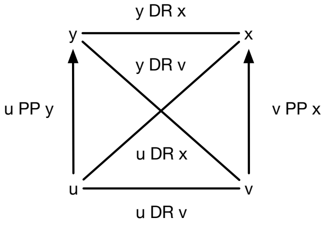

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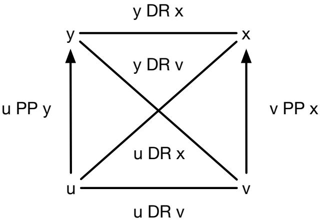

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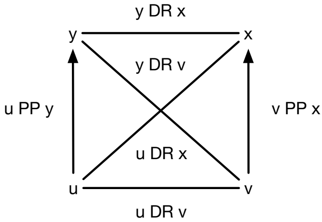

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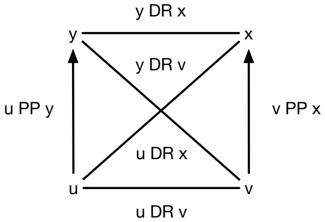

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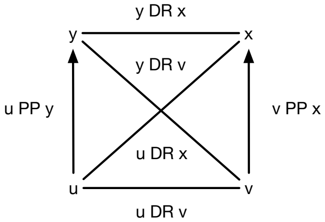

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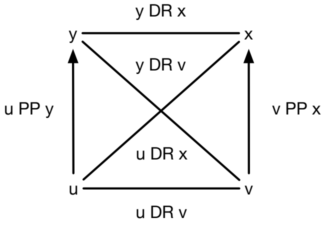

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Replace $\mathfrak{S}$ by $\mathfrak{S}'$. Task: Classify all first-order expansions $\mathfrak{B}$ of $\mathfrak{S}$.

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Canonisation

A permutation group G is called extremely amenable if every continuous action of G on a compact Hausdorff space has a fixed point.

$\mathrm{Aut}(\mathfrak{B})$: the automorphism group of $\mathfrak{B}$.

Lemma (Canonisation lemma; B.+Pinsker+Tsankov'11).

Suppose $\mathfrak{B}$ is ω-categorical and $\mathrm{Aut}(\mathfrak{B})$ is extremely amenable. Then for any $f\colon B^k \to B$, the set

$$\overline{\{(x_1,\dots,x_k) \mapsto a_0(f(a_1(x_1),\dots,a_k(x_k))) \mid a_0,a_1,\dots,a_k \in \mathrm{Aut}(\mathfrak{B})\}}$$

contains a function g that is canonical wrt $\mathfrak{B}$. (g is 'canonisation' of f)

Problem: The automorphism group of $\mathfrak{S}$ is not extremely amenable.

But (B.+Bodor'24): $\mathfrak{S}$ has homogeneous expansion $(\mathfrak{S};<)$ by a linear order $<$ such that $\mathrm{Aut}(\mathfrak{S};<)$ is extremely amenable.

Canonisation

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Write

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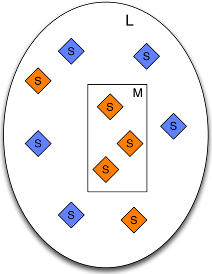

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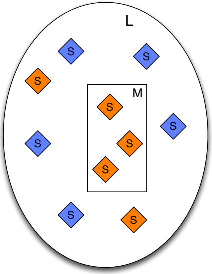

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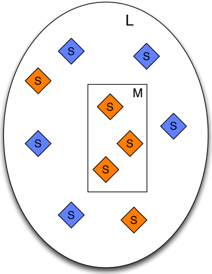

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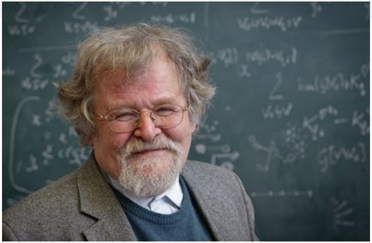

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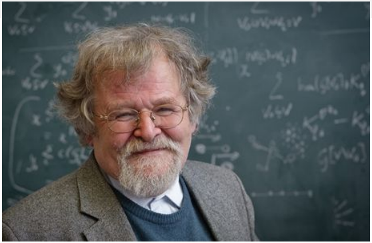

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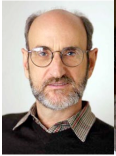

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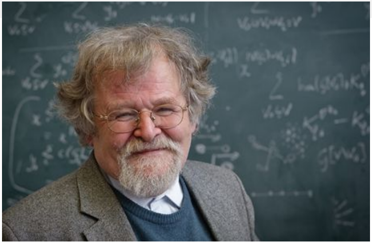

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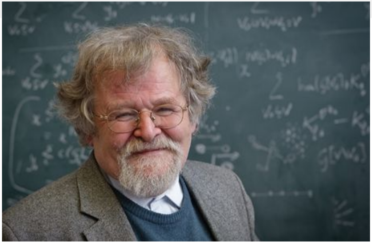

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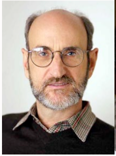

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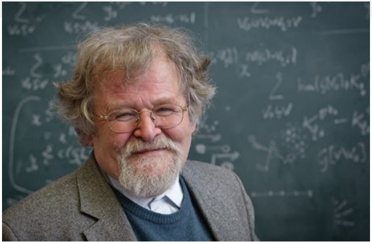

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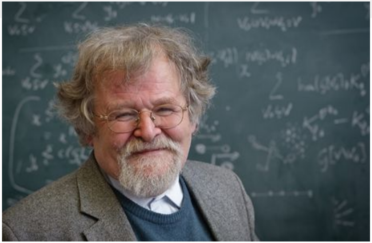

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Definition. $\mathcal{C}$: class of all expansions of structures from Age$(\mathfrak{S})$ by a binary relation $<$ which denotes a linear extension of PP.

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1. $\mathcal{C}$ is an amalgamation class.
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- Use the fact that the ordered countable atomless Boolean algebra $(A; \cap, \cup, \bar{\cdot}, 0, 1, <)$ is Ramsey (Graham+Rotschild'71,KPT'05).
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The set $\mathrm{Pol}(\mathfrak{B})$ of all polymorphisms of $\mathfrak{B}$ is a minion:
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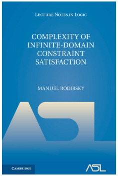
NP-hardness for Infinite-domain CSPs



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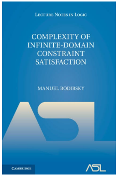

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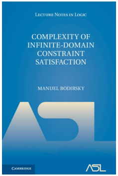

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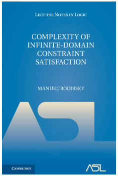

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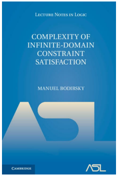

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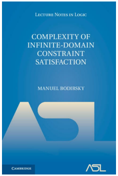

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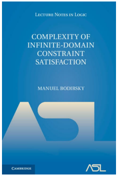

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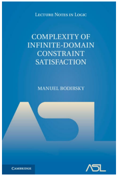

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Unique Interpolation Property

$\mathscr{C}$: set of all polymorphisms of $\mathfrak{B}$ that are canonical with respect to $\mathfrak{S}$.

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Appears implicitly in B.+Mottet'16 and explicitly in B.+Bodor'24.

Extension Lemma (B.+Bodor'24). If $\mu\colon \mathscr{C} \to \mathrm{Pol}(K_3)$ has the UIP, then μ can be extended to a uniformly continuous minor-preserving map from $\mathrm{Pol}(\mathfrak{B})$ to $\mathrm{Pol}(K_3)$.

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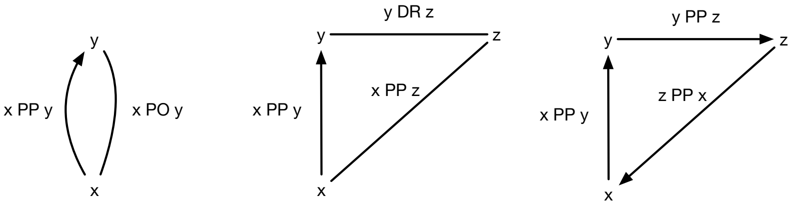
Open Problem



The structure $\mathfrak{S}$ is an example of a finitely bounded homogeneous structure.

Conjecture (B.+Pinsker'11): Every reduct $\mathfrak{B}$ of a finitely bounded homogeneous structure has a CSP which is in P or NP-complete.

Tractability Conjecture (B.+Pinsker'11, Barto+Kompatscher+Olšak+VanPham+Pinsker'16): If $\mathrm{Pol}(\mathfrak{B})$ has no uniformly continuous minor-preserving map to $\mathrm{Pol}(K_3)$, then $\mathrm{CSP}(\mathfrak{B})$ is in P.

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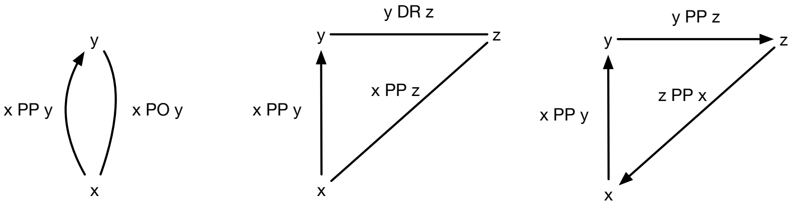

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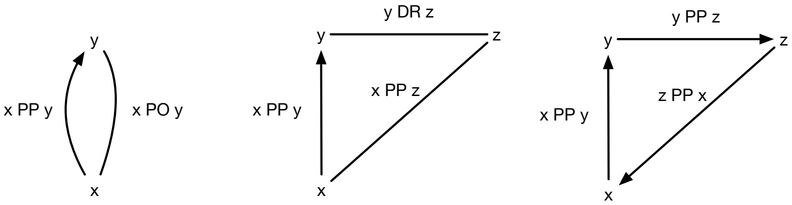

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