Model-theoretic Challenges in Constraint Satisfaction

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23.7.2024
Logik Oberseminar Freiburg

European Research Council
Established by the European Commission

ERC Synergy Grant POCOCOP (GA 101071674).

Overview

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3. The finite Ramsey expansion conjecture

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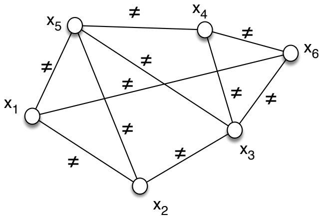

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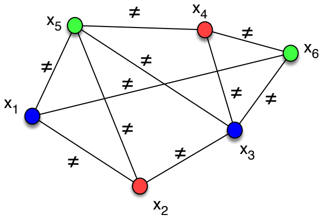

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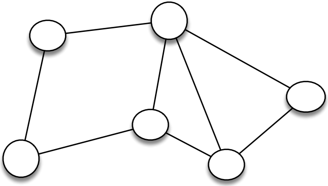

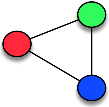

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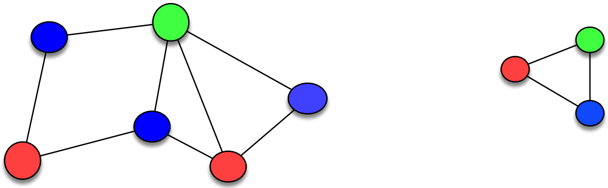

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Theorem. B.+Grohe'08: Every decision problem is equivalent to a CSP (under polynomial-time Turing reductions)

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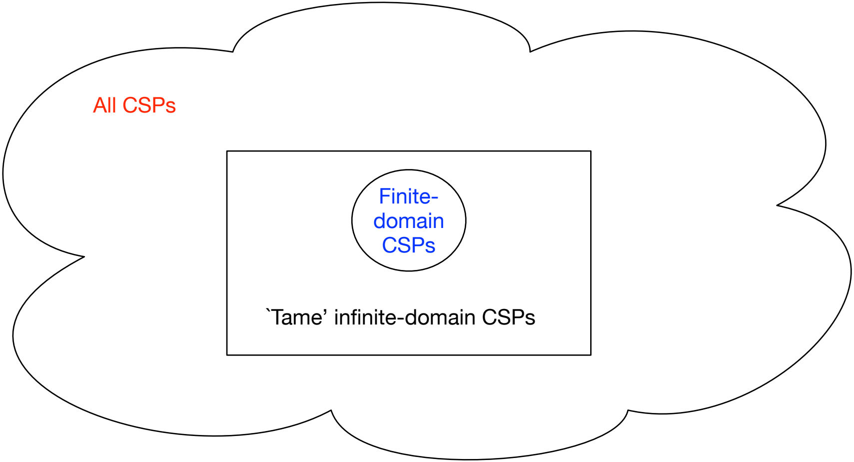

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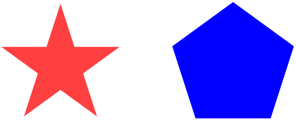

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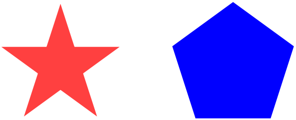

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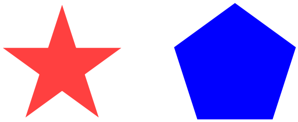

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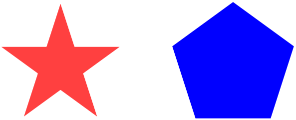

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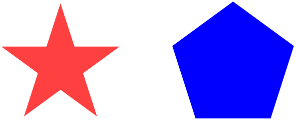

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- $\mathrm{Pol}(\mathfrak{B})$ is a clone: contains projections and closed under composition.

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Theorem (Bulatov'17, Zhuk'17). If $\text{Pol}(\mathfrak{C}, c_1, \ldots, c_n)$ does does not have a homomorphism to $\text{CSP}(K_3)$, then $\text{CSP}(\mathfrak{B})$ is in P.

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- CSP$\big(\mathbb{Q}; \{(x, y, z) \mid x > y \vee x > z\}\big)$ is in P.
- CSP$\big(\mathbb{Q}; \{(x, y, z) \mid x = y < z \vee y = z < x \vee z = x < y\}\big)$ is in P.

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Equivalently:
Then there are finitely many closed supergroups of Aut$(\mathfrak{C})$.

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$$\xi(g(f_1, \dots, f_n)) = \xi(g)(\xi(f_1), \dots, \xi(f_n))$$

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(2) Reconstruction Conjecture. Let $\mathfrak{A}$ and $\mathfrak{B}$ be reducts of structures that are homogeneous with finite relational signature. Then

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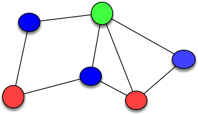

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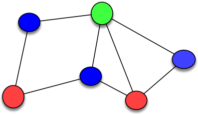

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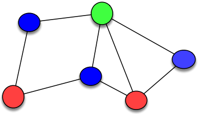

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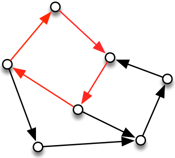

Countably Categorical Structures for MSO sentences

Theorem (B.+Knäuer+Rudolph'21).

Let $\mathfrak{B}$ be such that $\text{CSP}(\mathfrak{B})$ is in MSO. Then there exists an ω-categorical structure $\mathfrak{C}$ such that $\text{CSP}(\mathfrak{B}) = \text{CSP}(\mathfrak{C})$.

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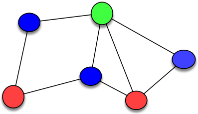

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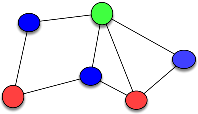

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Definition.

Write

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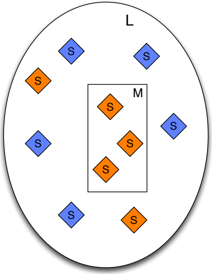

Proof: Ramsey Theory

For structures $\mathfrak{L}$ and $\mathfrak{S}$, write $\binom{\mathfrak{L}}{\mathfrak{S}}$ for the set of all embeddings of $\mathfrak{S} \hookrightarrow \mathfrak{L}$.

Definition.

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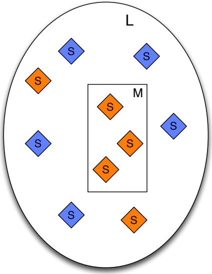

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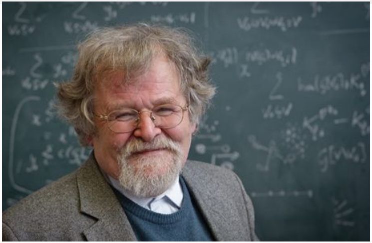

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A structure $\mathfrak{B}$ is Ramsey if
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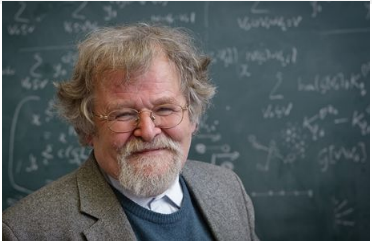

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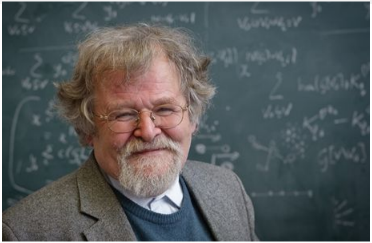

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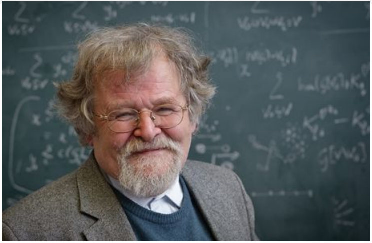

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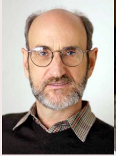
Theorem (Kechris, Pestov, Todorcevic'05).

A homogeneous structure $\mathfrak{B}$ is Ramsey if and only if $\text{Aut}(\mathfrak{B})$ is extremely amenable, i.e., every continuous action on a compact Hausdorff space has a fixed point.



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(3) Ramsey Expansion Conjecture. Every homogeneous structure with finite relational signature has a finite homogeneous Ramsey expansion.

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