The Complexity of Constraint Satisfaction with Semilinear Constraints

Manuel Bodirsky

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Outline

1. Constraint Satisfaction Problems: definition and examples

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2. Computational complexity of finite domain CSPs:

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   (a) Convex and Essentially Convex Semialgebraic Constraints

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For this audience: focus on geometric aspects.

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A computational problem:

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Question: Is there a satisfying assignment?

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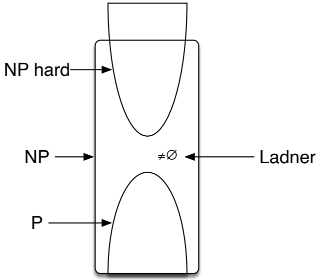
Central Question:
computational complexity of CSPs.



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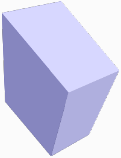

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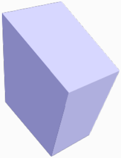

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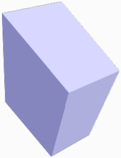

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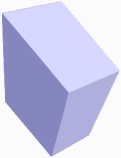

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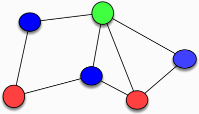

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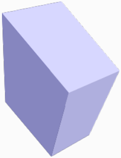

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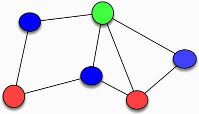

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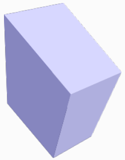

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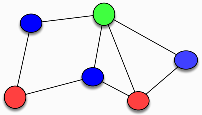

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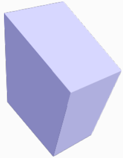

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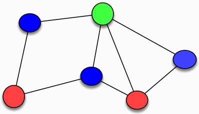

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Complexity: NP-complete

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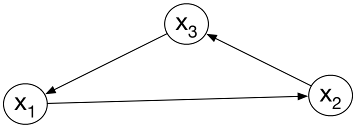
Example input: $x_1 < x_2 \wedge x_2 < x_3 \wedge x_3 < x_1$



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The CSPs in P have an elegant universal-algebraic characterisation.

Polymorphisms

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A function $f\colon D^k \to D$ preserves $R \subseteq D^m$ if for all $a^1, \dots, a^k \in R$
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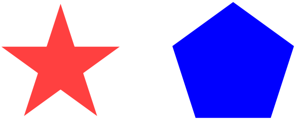

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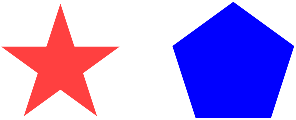

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f is called a polymorphism of $\mathfrak{B}$ if f preserves all relations of $\mathfrak{B}$.

Weak Near Unanimities

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Assume that $\mathfrak{B}$ has a finite domain D.

Theorem (Bulatov+Jeavons+Krokhin'05, Maroti+McKenzie'08).

Let $\mathfrak{B}$ be a finite structure. Then $\mathfrak{B}$ has a weak near unanimity polymorphism, that is, a polymorphism f of arity $k \geq 2$ such that for all elements x, y of $\mathfrak{B}$

$$f(y, x, \ldots, x) = f(x, y, \ldots, x) = \cdots = f(x, \ldots, x, y) \; ,$$

or CSP$(\mathfrak{B})$ is NP-hard.

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is a weak near unanimity polymorphism of $(\mathbb{Q}; <)$.

If CSP$(\mathfrak{B})$ is in P, then $\mathfrak{B}$ must have 'Higher-dimensional symmetry'

Finite-Domain Tractability Theorem

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Theorem (B.+Grohe'08). For every computational problem $\mathcal{C}$ there exists a structure $\mathfrak{B}$ such that $\mathcal{C}$ is equivalent (under polynomial-time Turing reductions) to CSP$(\mathfrak{B})$.

The Rational Numbers

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Big challenge:
Classify the complexity of $\text{CSP}(\mathbb{Q}; R_1, \ldots, R_\ell)$ for all semilinear relations $R_1, \ldots, R_\ell$.

The Rational Numbers

Definition.

$R \subseteq \mathbb{Q}^k$ is semilinear if R has a first-order definition in $(\mathbb{Q}; +, 1, \leq)$.

$\forall, \exists, \wedge, \vee, \neg$

Ferrante and Rackoff'75: A relation is semilinear if and only if it is a finite intersection of finite unions of (open or closed) linear half spaces.

Big challenge:
Classify the complexity of CSP$(\mathbb{Q}; R_1, \dots, R_\ell)$ for all semilinear relations $R_1, \dots, R_\ell$.

Remark. Exists solution over $\mathbb{Q} \Leftrightarrow$ exists solution over $\mathbb{R}$.

Examples

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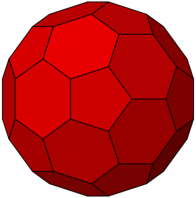

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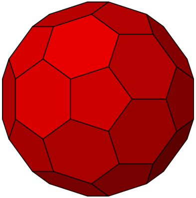

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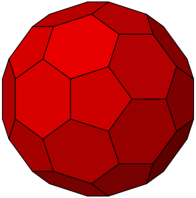

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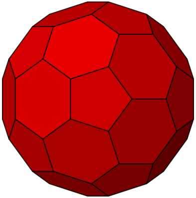

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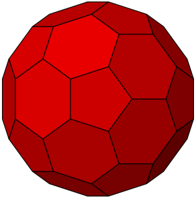

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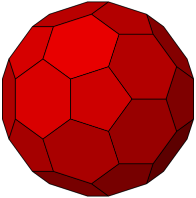

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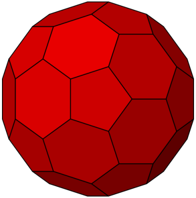

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   In P (Jonsson+Bäckström'98).

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$\mathfrak{B} = (\mathbb{Q}; R_1, \dots, R_\ell)$ such that $R_1, \dots, R_\ell$ are first-order definable over $(\mathbb{Q};<)$.
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- A linear inequality $a_1 x_1 + \cdots + a_m x_m \leq a_0$
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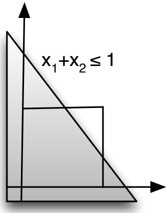
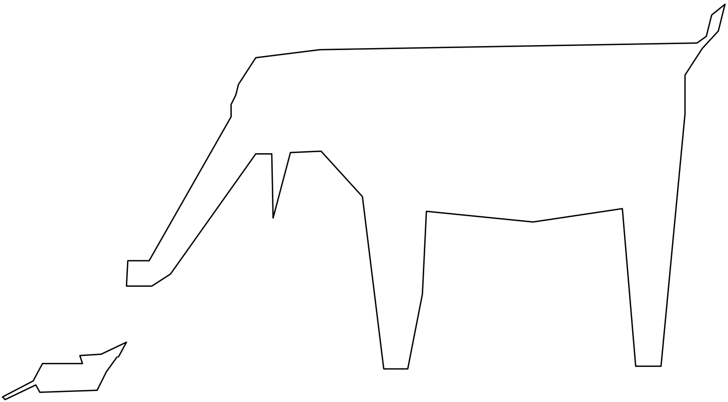


Example

The enclosed subset of $\mathbb{R}^2$ is preserved by max.



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where $\prec_i \in \{\leq, <\}$, $c_1, \dots, c_m \in \mathbb{Q}$, $\bar{a}_1, \dots, \bar{a}_m \in \mathbb{Q}^n$, and there is a $k \leq n$ such that $\bar{a}_{i,j} \geq 0$ for all i and $j \neq k$.

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- R is projection of solution space of an instance of $\text{CSP}(\mathbb{Q}; <, \{1\}, \{-1\}, S_1, S_2, S_3)$ where

  $$S_1 := \{(x, y) : 2x \leq y\},$$
  $$S_2 := \{(x, y, z) : x \leq y + z\}$$
  $$S_3 := \{(x, y, z) : x \leq y \vee x \leq z\}$$

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And/Or Precedence Constraints (Möhring+Stork+Skutella'04)
aka Max-Atom Problem (Bezem,Nieuwenhuis+Rodriguez-Charbonell'08)
equiv to Emptyness of Tropical Polyhedra (Akian+Gaubert+Guterman'11)
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- MSS'04, BNR'08, AGG'11: in coNP.

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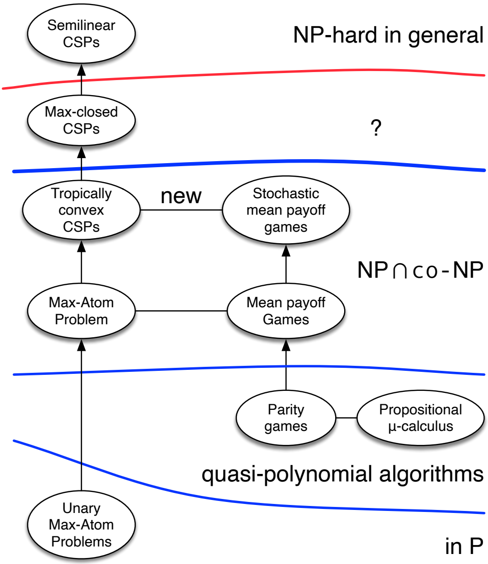
Then $\mathrm{CSP}(\mathbb{Q}; S_1, \ldots, S_\ell)$ is in NP $\cap$ coNP.

Overview State of the Art



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where $\prec_i \in \{\leq, <\}$, $c_1, \dots, c_m \in \mathbb{Q}$, $\bar{a}_1, \dots, \bar{a}_m \in \mathbb{Q}^n$, and there is a $k \leq n$ such that $\bar{a}_{i,j} \geq 0$ for all i and $j \neq k$ and $\sum_j a_{i,j} = 0$ for all i.

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- R has a primitive positive definition in $(\mathbb{Q}; <, S_3, T_1, T_{-1}, S_4)$ where

$$T_{\pm 1} := \{(x, y) : x \leq y \pm 1\}$$
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Consequence: satisfiability of P is in NP $\cap$ coNP.

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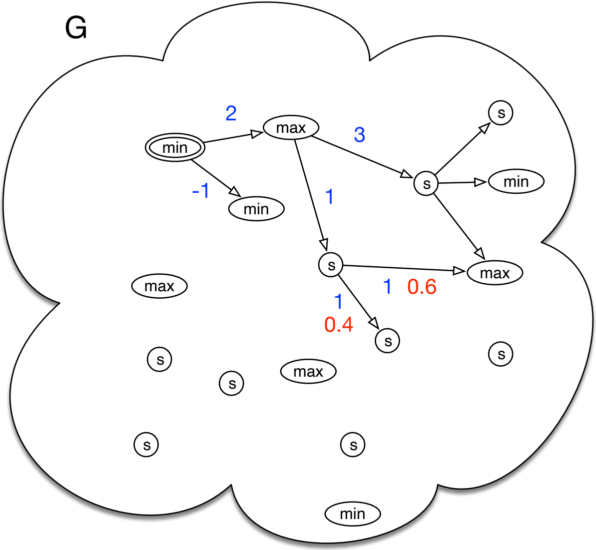

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