Forbidden Tournaments and the Orientation (Completion) Problem

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Oriented expressions of graph classes

Example 1 (Robbins 1939)

A graph G is 2-edge-connected if and only if it admits a strongly connected orientation.

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Example 2 (Standard definition)

A graph G is a comparability graph if and only if it admits a transitive orientation.

Oriented expressions of graph classes

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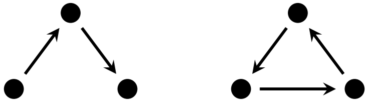
Example 3 (Roy-Gallai-Hasse-Vitaver Theorem)

A graph G is k-colourable if and only if it admits an orientation with no directed walk of length k.

Oriented expressions of graph classes

Example 2 (Standard definition)

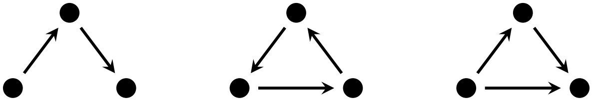
A graph G is a comparability graph if and only if it admits an $\mathcal{F}$-free orientation orientation.



Oriented expressions of graph classes

Example 3 ($k = 2$ of RGHV-Theorem)

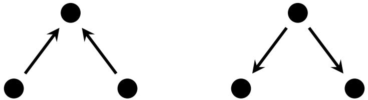
A graph G is a bipartite graph if and only if it admits an $\mathcal{F}$-free orientation.



Oriented expressions of graph classes

Example 4 (Skrien 1982)

A connected graph G is a proper circular-arc graph if and only if it admits an $\mathcal{F}$-free orientation.

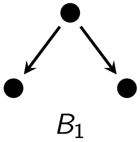


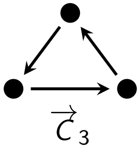
Oriented expressions of graph classes

Characterization Problem

Given a finite set of oriented graphs $\mathcal{F}$ characterize the class of graphs that admit an $\mathcal{F}$-free orientation (e.g., list their minimal obstructions).

- Orientations of P_3 (Skrien 1982).
- Oriented graphs on 3 vertices (G.P. and Hernández-Cruz 2021).
- Open cases: $\mathcal{F} = \{B_1\}$ and $\mathcal{F} = \{B_1, \overrightarrow{C}_3\}$.



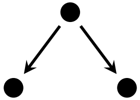


Oriented expressions of graph classes

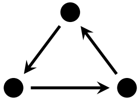
Complexity Problem

Given a finite set of oriented graphs $\mathcal{F}$, determine the complexity of deciding if an input graph G admits an $\mathcal{F}$-free orientation?

- In P when $\mathcal{F}$ is a set of oriented graphs on 3 vertices (Urrutia and Gavril 1992, Bang-Jensen and Gutin 2007, G.P. and Hernández-Cruz 2021).
- Open case: $\mathcal{F} = \{B_1, \overrightarrow{C}_3\}$.



B_1



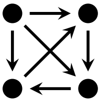
$\overrightarrow{C}_3$

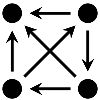
Oriented expressions of graph classes

Complexity Problem (completion version)

Given a finite set of oriented graphs $\mathcal{F}$, determine the complexity of deciding if an input partially oriented graph G can be completed to an $\mathcal{F}$-free oriented graph?

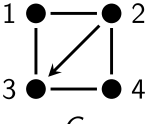
- Orientations of P_3 always in P (Bang-Jensen, Huang, Zhu, 2017).
- T_3-free orientation completion problem in P.
- (Bang-Jensen, Huang, Zhu) NP-complete for:





Complexity Classification

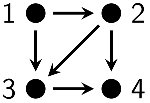
Complexity Classification



G

Code orientation completions of G as solutions to the sys. lin. eq. over $\mathbb{Z}_2$

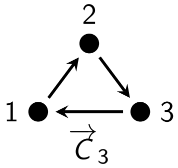
$$x_{ij} + x_{ji} = 0 \text{ for } ij \in U$$
$$x_{ij} = 1 \text{ for } ij \in E$$



G'

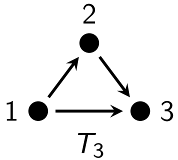
$$x_{12} = 1,\ x_{13} = 1,\ x_{23} = 1,\ x_{24} = 1,\ x_{34} = 1$$
$$x_{21} = 0,\ x_{31} = 0,\ x_{32} = 0,\ x_{42} = 0,\ x_{43} = 0$$

Complexity Classification



For each triangle i, j, k the following equality holds:

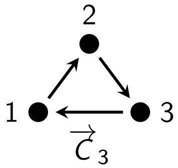
$$x_{ij} + x_{jk} = 0.$$



There exists a triangle i, j, k such that the following equality holds:

$$x_{ij} + x_{jk} = 1 \text{ for instance } x_{23} + x_{31} = 1.$$

Complexity Classification



For each triangle i, j, k the following equality holds:

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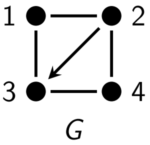
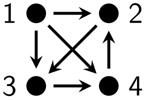


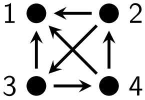
Code T_3-free orientation completions of G as solutions to

$$\begin{aligned} x_{ij} + x_{ji} &= 0 \text{ for } ij \in U \\ x_{ij} &= 1 \text{ for } ij \in E \\ x_{ij} + x_{jk} &= 0 \text{ for } ijk \in T \end{aligned}$$

Complexity Classification



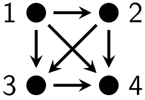
C_3^-



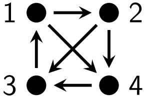
C_3^+

For each i, j, k, l in C_3^- and in C_3^+

$$x_{ij} + x_{jk} + x_{kl} + x_{li} = 0.$$



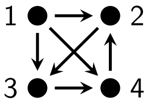
T_4



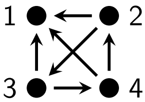
TC_4

$$x_{12} + x_{23} + x_{34} + x_{41} = 1 \text{ in } T_4$$
$$x_{12} + x_{23} + x_{34} + x_{41} = 1 \text{ in } TC_4$$

Complexity Classification



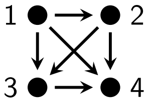
C_3^-



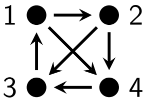
C_3^+

$\mathcal{F}$-free orientation completions of G

$$x_{ij} + x_{ji} = 0 \text{ for } ij \in U$$
$$x_{ij} = 1 \text{ for } ij \in E$$
$$x_{ij} + x_{jk} + x_{kl} + x_{li} = 0 \text{ for } ijkl \in K_4(G).$$



T_4



TC_4

$$x_{12} + x_{23} + x_{34} + x_{41} = 1 \text{ in } T_4$$
$$x_{12} + x_{23} + x_{34} + x_{41} = 1 \text{ in } TC_4$$

Complexity Classification

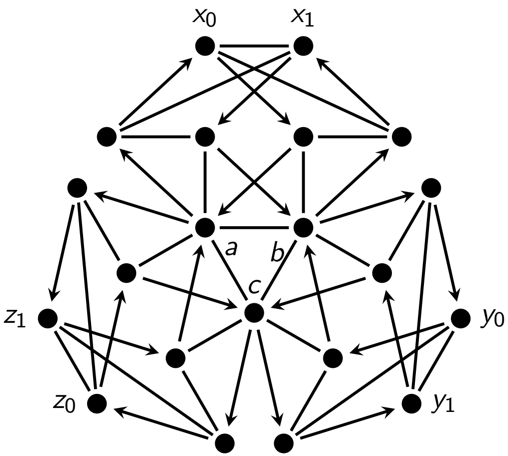
NP-hard example

The $\overrightarrow{C}_3$-free orientation completion problem in NP-complete, via reduction from not-all-equal 3-sat

not-all-equal 3-**sat problem**
Input: $(x_1 \vee y_1 \vee z_1) \wedge \cdots \wedge (x_k \vee y_k \vee z_k)$
Solution: a function $f \colon V \to \{0,1\}$ such that

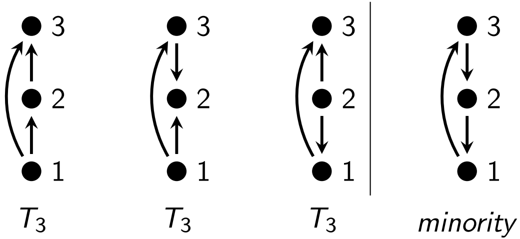
$$(f(x_i), f(y_i), f(z_i)) \in \{0,1\}^3 \setminus \{(0,0,0),(1,1,1)\}.$$

Complexity Classification



Gadget for $(x \vee y \vee z)$ where
$f(x) = 1$ iff (x_0, x_1) in $\overrightarrow{C}_3$-free orientation completion.

Complexity Classification



- $\overrightarrow{C}_3$-free tournaments **are not** *preserved* by the *minority* operation.
- T_3-free tournaments **are** *preserved* by the *minority* operation.

Complexity Classification

Theorem (Bodirsky, G.P., 23+)

For every finite set of finite tournaments $\mathcal{F}$ one of the following cases holds.

1. $\mathcal{F}_f$ is preserved by the minority operation. In this case, the $\mathcal{F}$-free orientation completions of a partially oriented graph G correspond to the solution space of a system of linear equations over $\mathbb{Z}_2$.
2. Otherwise, $\mathcal{F}$-free orientation completion problem is NP-complete.

In the first case, the $\mathcal{F}$-free orientation completion problem is in P.

Complexity Classification

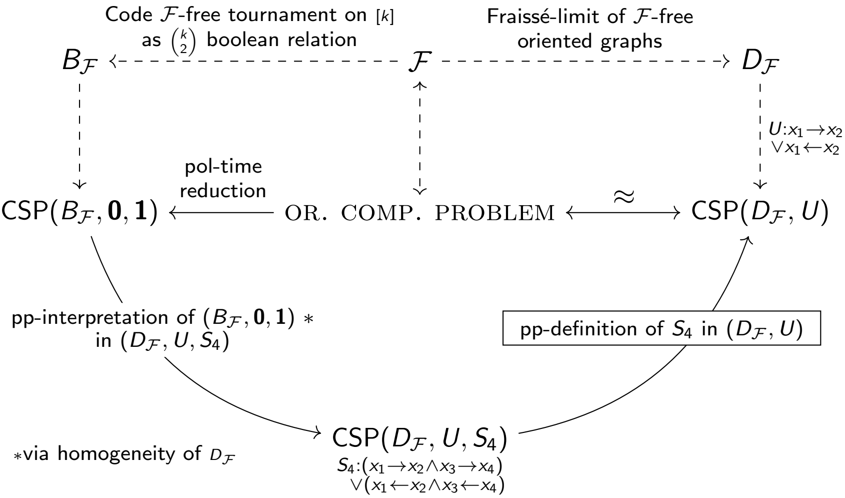
Theorem (Bodirsky, G.P., 23+)

For every finite set of finite tournaments $\mathcal{F}$ one of the following cases holds.

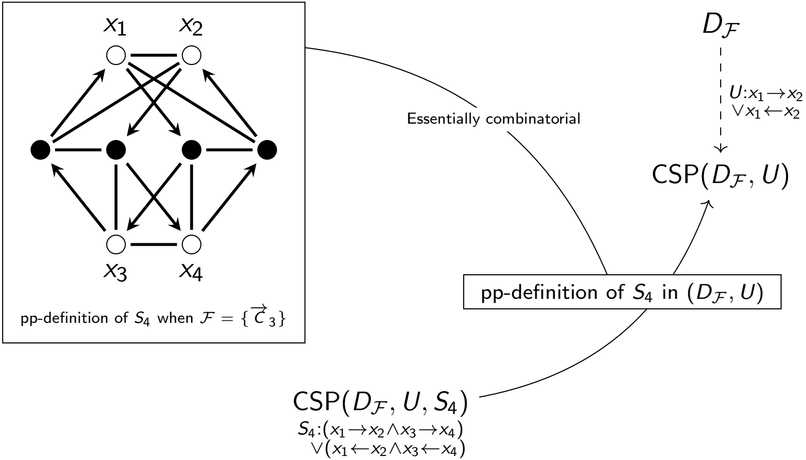
1. $\mathcal{F}$ contains no transitive tournament. In this case, every graph admits an $\mathcal{F}$-free orientation.
2. $\mathcal{F}_f$ is preserved by the minority operation. In this case, the $\mathcal{F}$-free orientations of a graph G correspond to the solution space of a system of linear equations over $\mathbb{Z}_2$.
3. Otherwise, $\mathcal{F}$-free orientation problem is NP-complete.

In cases 1 and 2, the $\mathcal{F}$-free orientation problem is in P.

Proof overview



Proof overview



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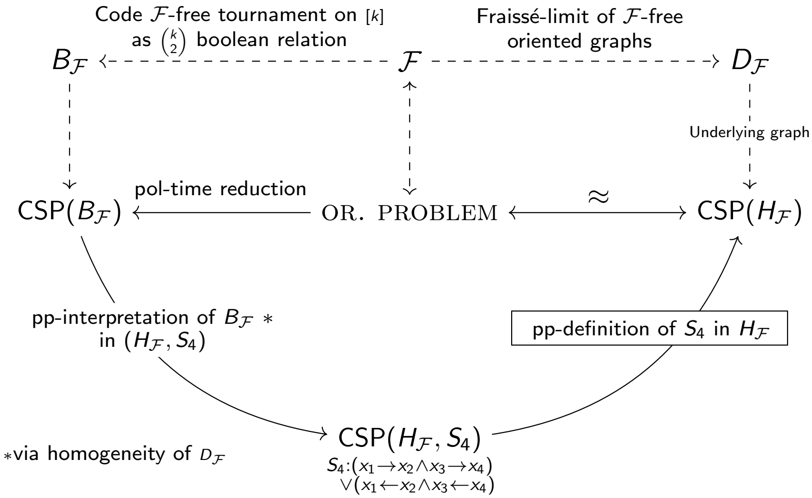

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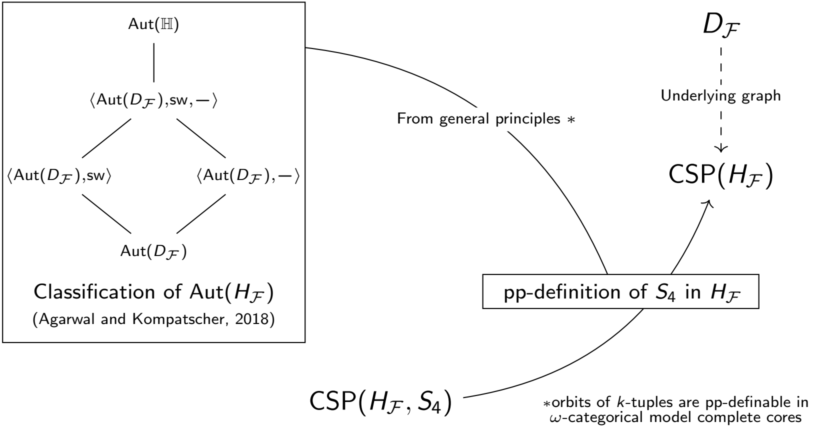

Thank you!

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