Clones on 3 elements: A New Hope

Dmitriy Zhuk
zhuk.dmitriy@gmail.com

Charles University
Prague

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Outline

Clones!

I. The Phantom Problem!

II. Attack the Continuum!

III. Revenge of the Continuum.

IV. A New Hope!

V. Continuum Strikes Back

VI. Return of Beautiful Clones

Clones!

Clone is a set of operations closed under composition and containing all projections.

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Examples of clones

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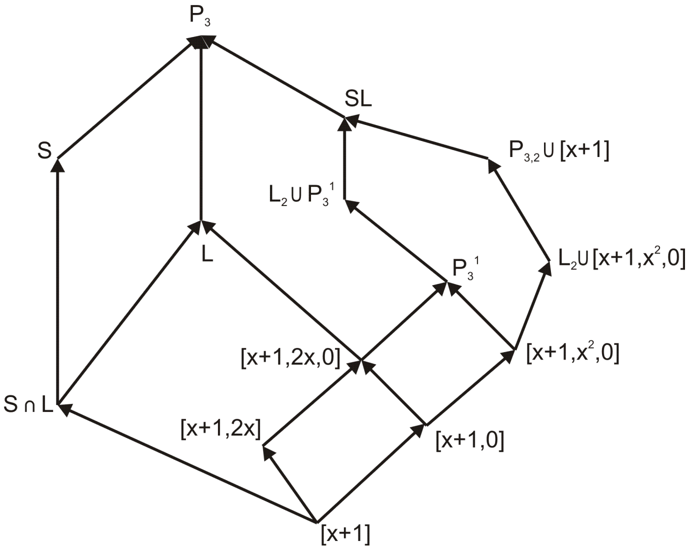
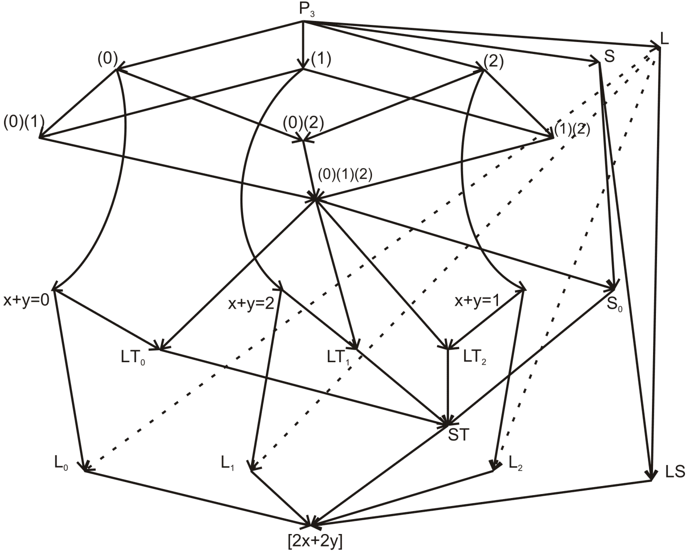
Clones ordered by inclusion form a lattice.

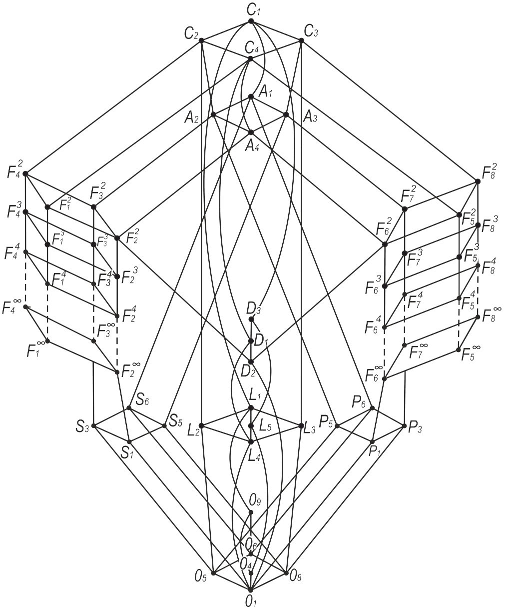
The Lattice of Clones containing $x + 1$ on $\{0, 1, 2\}$



The Lattice of Clones containing $2x + 2y$ on $\{0, 1, 2\}$



The lattice of all clones on two elements(for $|A| = 2$)



Emil Post (1921, 1941)

Galois connection

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Relational clone is a set of relations closed under pp-formulas

$$\exists y_1 \dots \exists y_s\ R_1(\dots) \wedge \dots \wedge R_m(\dots)$$

and containing $=$ and $\varnothing$.

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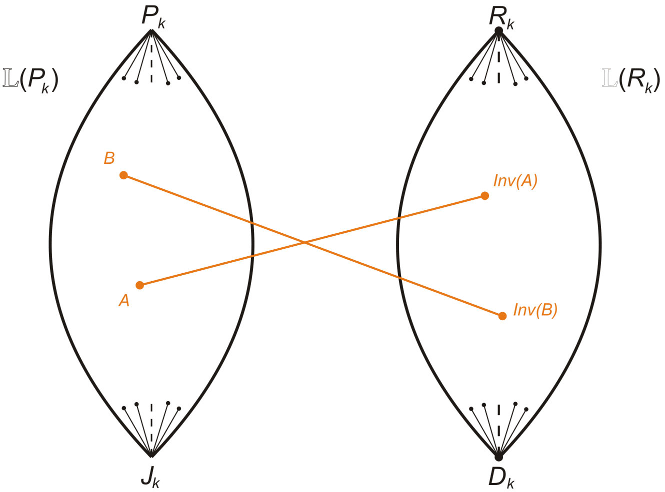
Theorem [Bodnarchuk, Kaluzhnin, Kotov, Romov, Geiger, 1969]

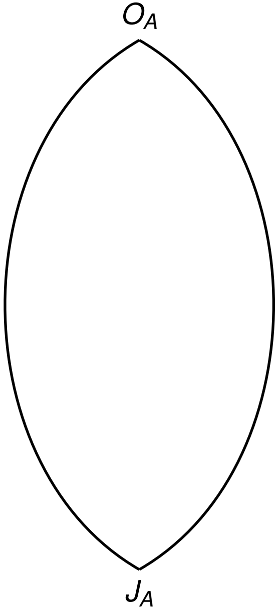
- $\mathrm{Pol}(\mathrm{Inv}(\mathcal{C})) = \mathcal{C}$ for any clone C.
- $\mathrm{Inv}(\mathrm{Pol}(\mathcal{R})) = \mathcal{R}$ for any relational clone $\mathcal{R}$.
- $\mathrm{Pol}(\mathrm{Inv}(\mathcal{F})) = \mathrm{Clo}(\mathcal{F})$.
- $\mathrm{Inv}(\mathrm{Pol}(\mathcal{S})) = \mathrm{RelClo}(\mathcal{S})$.
- $\mathcal{R}_1 \subseteq \mathcal{R}_2 \Rightarrow \mathrm{Pol}(\mathcal{R}_1) \supseteq \mathrm{Pol}(\mathcal{R}_2)$.
- $\mathcal{C}_1 \subseteq \mathcal{C}_2 \Rightarrow \mathrm{Inv}(\mathcal{C}_1) \supseteq \mathrm{Inv}(\mathcal{C}_2)$.

Galois connection



I. The Phantom Problem!

For $|A| > 2$



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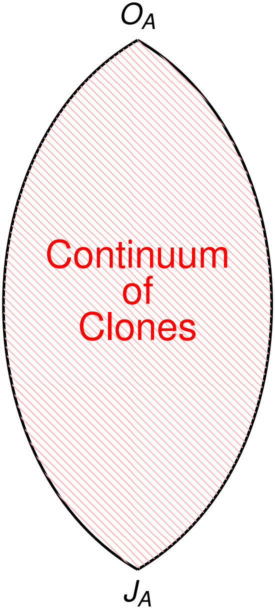

- There exists a continuum of clones for $|A| > 2$ (Ju. I. Janov, A. A. Muchnik, 1959)

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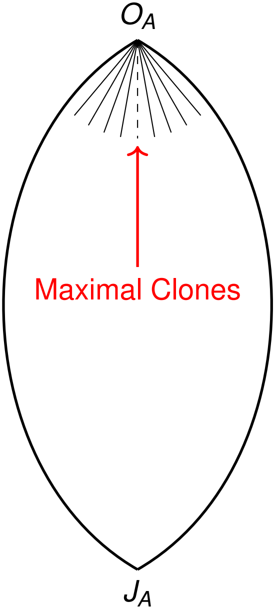

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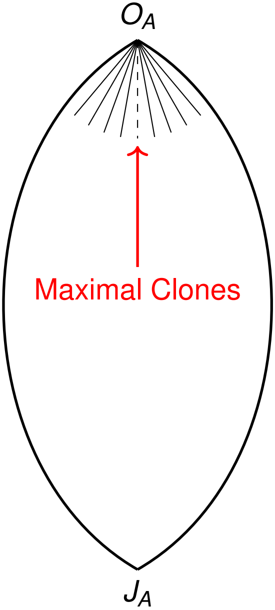

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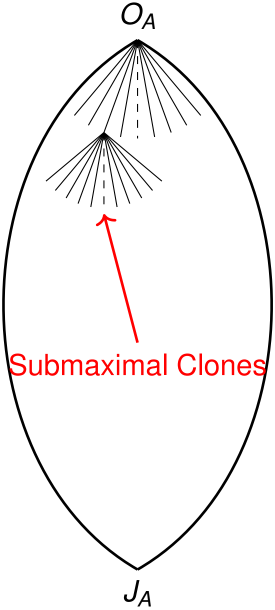

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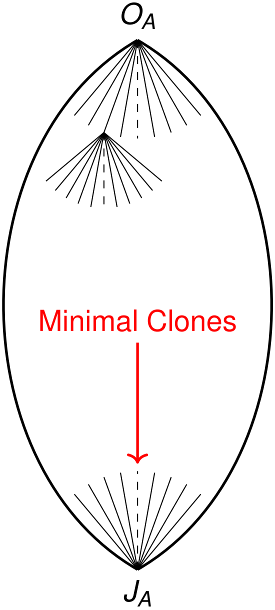

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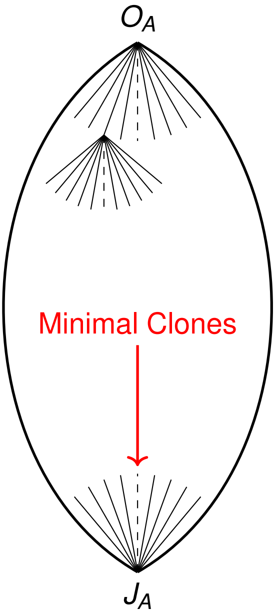

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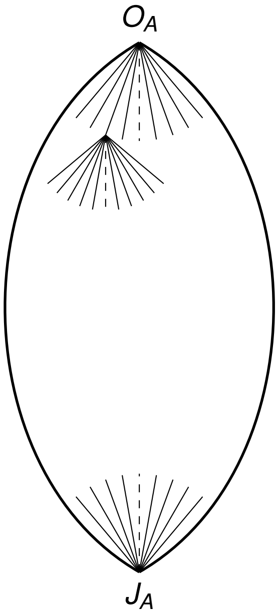

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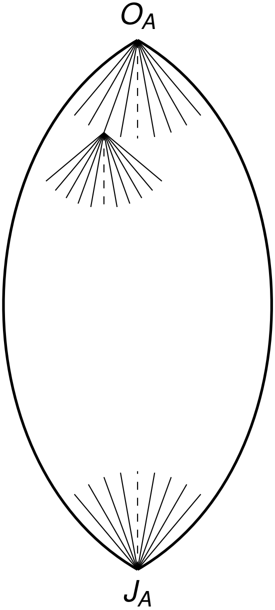

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- All minimal clones for $|A| = 4$ were found (Karsten Schölzer, 2012)

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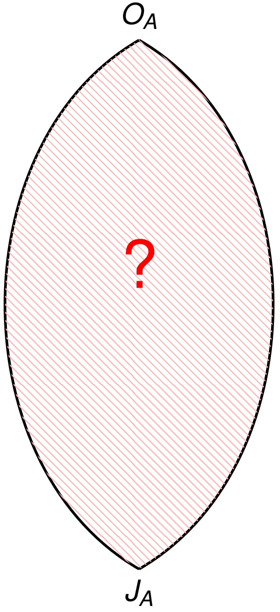

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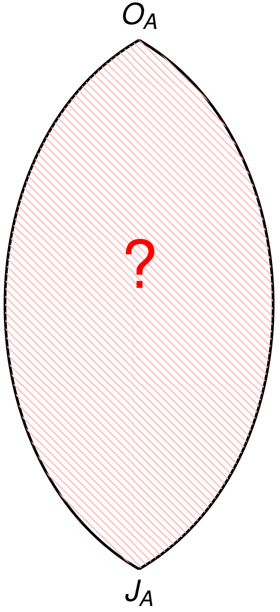
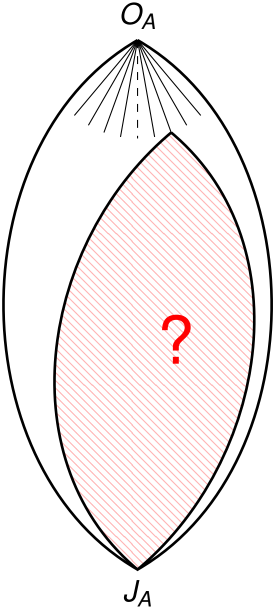


- Can we describe a significant part of the lattice?

For $|A| > 2$



- Can we describe all subclones of a maximal clone?

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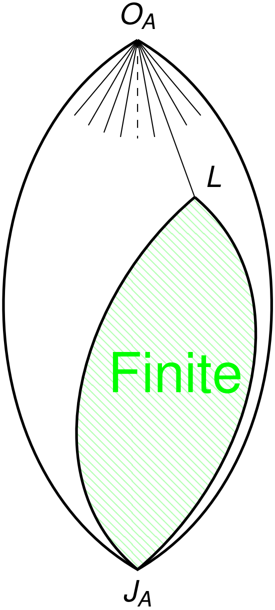

- Can we describe all subclones of a maximal clone?

- For the maximal clone of linear operations the lattice of subclones is finite and known ($|A|$ is a prime number) (A. A. Salomaa, 1964)

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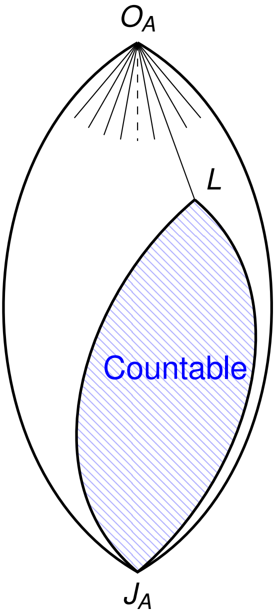

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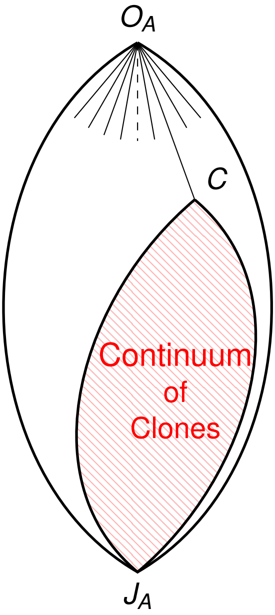

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- For the maximal clone of quasi-linear operations the lattice of subclones is countable but not known (if $|A|$ is a power of a prime number)
- For all other maximal clones the lattice of subclones is uncountable (J. Demetrovics, L. Hannak, S. S. Marchenkov, 1983)

II. Attack the Continuum!

$A = \{0, 1, 2\}$

Clone of Self-Dual Operations

$$\mathcal{C}_3 = \mathrm{Pol}\begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \end{pmatrix}$$

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- There exist continuum clones of self-dual operations (S.S. Marchenkov, 1983).

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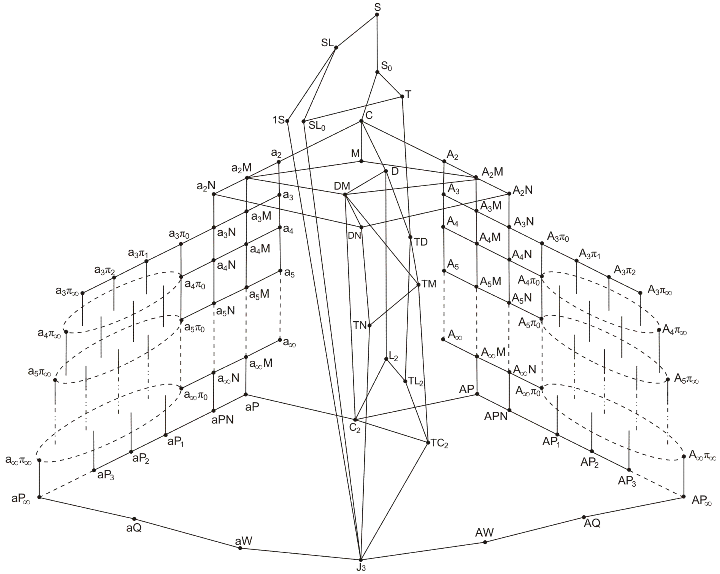
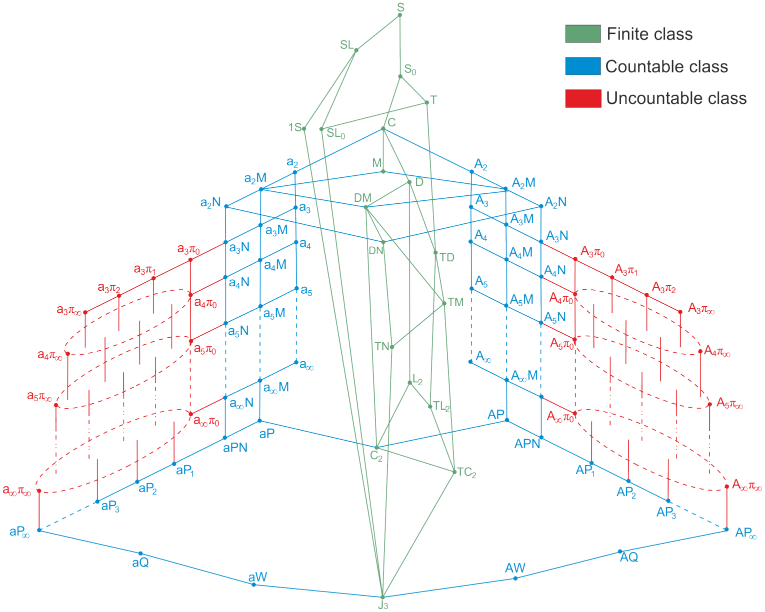
- There exist continuum clones of self-dual operations (S.S. Marchenkov, 1983).

(D. Zhuk, 2010)

A complete description of clones of self-dual operations on three elements


S
SL
S_0
T
1S
SL_0
C
a_2
M
A_2
a_2M
D
A_2M
a_2N
DM
A_2N
a_3
A_3
a_3M
A_3M
$a_3\pi_0$
a_3N
DN
a_4
A_4
A_3N
$A_3\pi_0$
$a_3\pi_1$
a_4M
TD
A_4M
$A_3\pi_1$
$a_3\pi_2$
a_4N
A_5
A_4N
$A_3\pi_2$
$a_3\pi_\infty$
$a_4\pi_0$
a_5
TM
A_5M
$A_4\pi_0$
$A_3\pi_\infty$
a_5M
a_5N
A_5N
$a_4\pi_\infty$
$a_5\pi_0$
TN
$A_5\pi_0$
$A_4\pi_\infty$
a_∞
A_∞
$a_5\pi_\infty$
$a_\infty M$
L_2
$A_\infty M$
$A_\infty N$
$A_5\pi_\infty$
$a_\infty N$
TL_2
$a_\infty\pi_0$
AP
$A_\infty\pi_0$
aP
aPN
APN
C_2
aP_1
TC_2
AP_1
$a_\infty\pi_\infty$
aP_2
AP_2
$A_\infty\pi_\infty$
aP_3
AP_3
aP_∞
AP_∞
aQ
AQ
aW
AW
J_3


Finite class
Countable class
Uncountable class
S
SL
S_0
T
1S
SL_0
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DN
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L_2
TL_2
C_2
TC_2
J_3
a_2
a_2M
a_2N
a_3
a_3M
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a_4
a_4M
a_4N
a_5
a_5M
a_5N
a_∞
$a_\infty M$
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aP
aPN
aP_1
aP_2
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aQ
aW
$a_3\pi_0$
$a_3\pi_1$
$a_3\pi_2$
$a_3\pi_\infty$
$a_4\pi_0$
$a_4\pi_\infty$
$a_5\pi_0$
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$a_\infty\pi_0$
$a_\infty\pi_\infty$
A_2
A_2M
A_2N
A_3
A_3M
A_3N
A_4
A_4M
A_4N
A_5
A_5M
A_5N
A_∞
$A_\infty M$
AP
APN
AP_1
AP_2
AP_3
AP_∞
AQ
AW
$A_3\pi_0$
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$A_4\pi_0$
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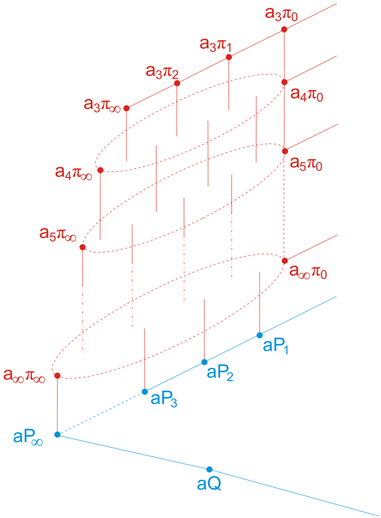
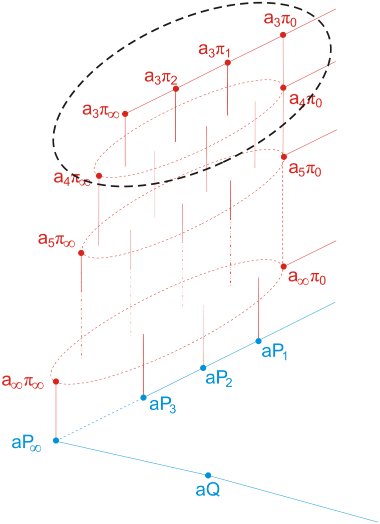
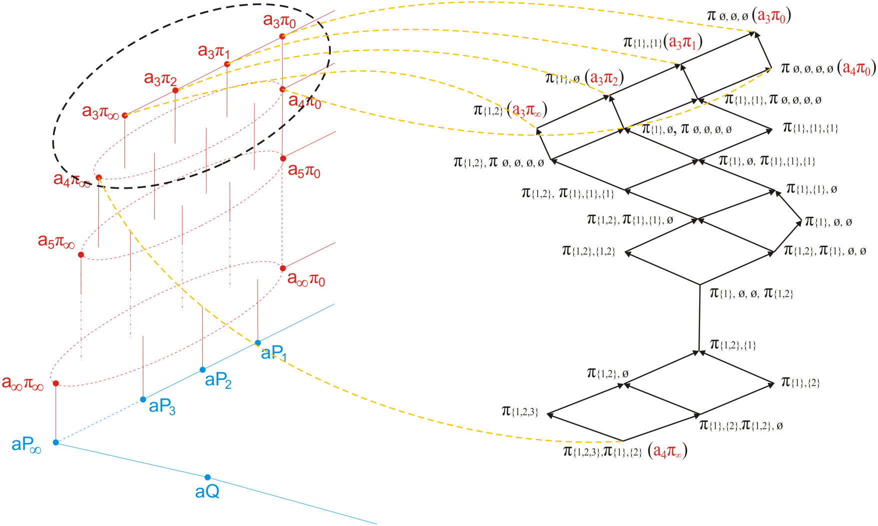
$A_5\pi_\infty$
$A_\infty\pi_0$
$A_\infty\pi_\infty$


a3π0
a3π1
a3π2
a3π∞
a4π0
a5π0
a4π∞
a5π∞
a∞π0
a∞π∞
aP1
aP2
aP3
aP∞
aQ



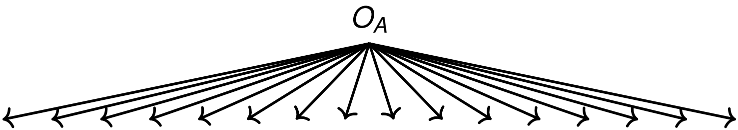
a3π0
a3π1
a3π2
a3π∞
a4π0
a4π∞
a5π0
a5π∞
a∞π0
a∞π∞
aP1
aP2
aP3
aP∞
aQ



$a_3\pi_0$
$a_3\pi_1$
$a_3\pi_2$
$a_3\pi_\infty$
$a_4\pi_0$
$a_4\pi_\infty$
$a_5\pi_0$
$a_5\pi_\infty$
$a_\infty\pi_0$
$a_\infty\pi_\infty$
aP_1
aP_2
aP_3
aP_∞
aQ
$\pi_{\emptyset,\emptyset,\emptyset}$ $(a_3\pi_0)$
$\pi_{\{1\},\{1\}}$ $(a_3\pi_1)$
$\pi_{\emptyset,\emptyset,\emptyset,\emptyset}$ $(a_4\pi_0)$
$\pi_{\{1\},\emptyset}$ $(a_3\pi_2)$
$\pi_{\{1\},\{1\}}, \pi_{\emptyset,\emptyset,\emptyset,\emptyset}$
$\pi_{\{1,2\}}$ $(a_3\pi_\infty)$
$\pi_{\{1\},\emptyset}, \pi_{\emptyset,\emptyset,\emptyset,\emptyset}$
$\pi_{\{1\},\{1\},\{1\}}$
$\pi_{\{1,2\}}, \pi_{\emptyset,\emptyset,\emptyset,\emptyset}$
$\pi_{\{1\},\emptyset}, \pi_{\{1\},\{1\},\{1\}}$
$\pi_{\{1,2\}}, \pi_{\{1\},\{1\},\{1\}}$
$\pi_{\{1\},\{1\},\emptyset}$
$\pi_{\{1,2\}}, \pi_{\{1\},\{1\},\emptyset}$
$\pi_{\{1\},\emptyset,\emptyset}$
$\pi_{\{1,2\},\{1,2\}}$
$\pi_{\{1,2\}}, \pi_{\{1\},\emptyset,\emptyset}$
$\pi_{\{1\},\emptyset,\emptyset}, \pi_{\{1,2\}}$
$\pi_{\{1,2\},\{1\}}$
$\pi_{\{1,2\},\emptyset}$
$\pi_{\{1\},\{2\}}$
$\pi_{\{1,2,3\}}$
$\pi_{\{1\},\{2\}},\pi_{\{1,2\},\emptyset}$
$\pi_{\{1,2,3\}},\pi_{\{1\},\{2\}}$ $(a_4\pi_\infty)$

III. Revenge of the Continuum.

Clones with a majority operation on 3 elements



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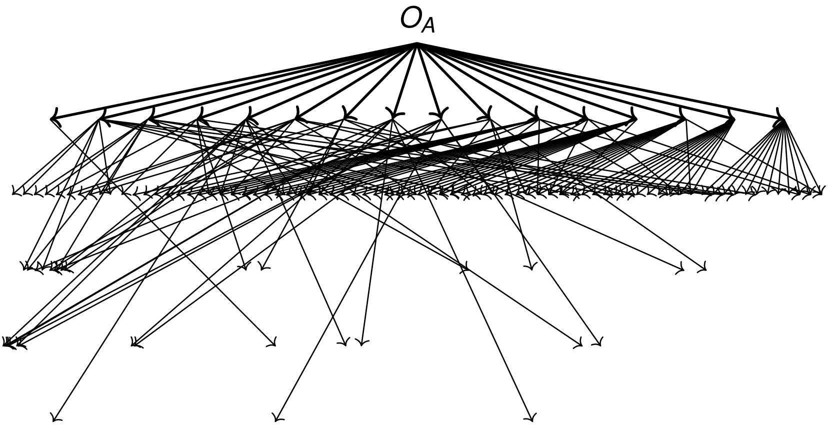

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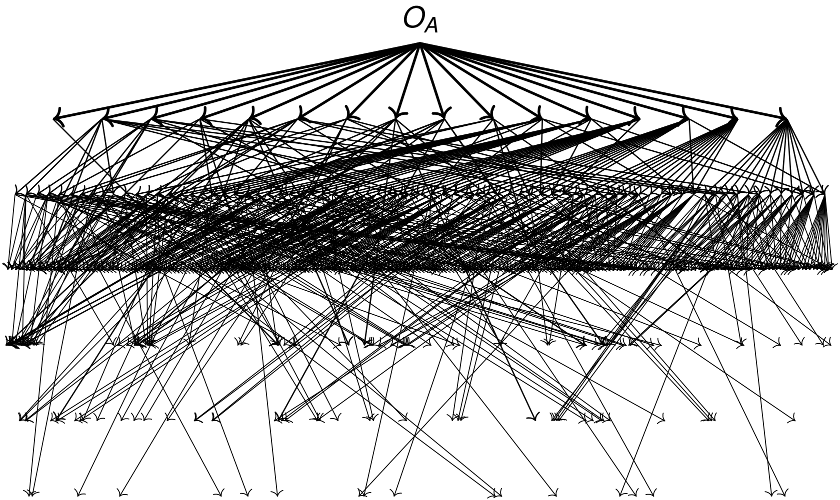

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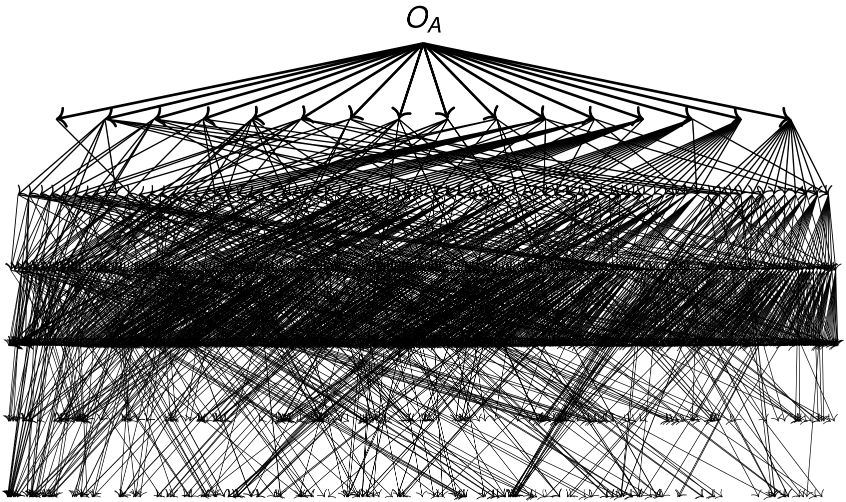
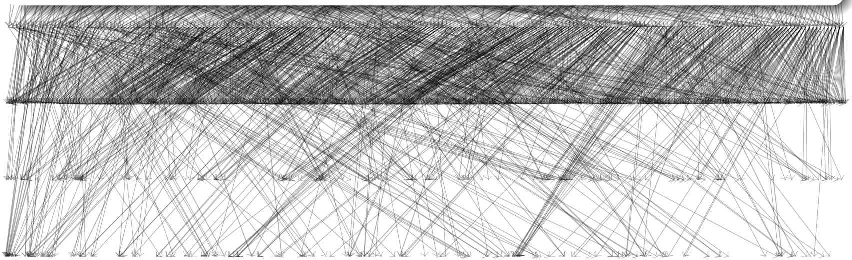
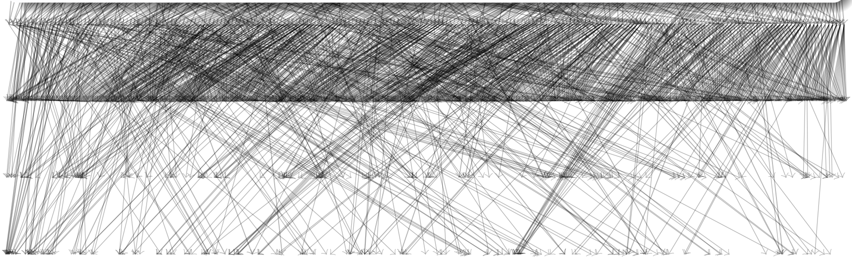


Computer Calculations [Moiseev, Zhuk, 2017]



Computer Calculations [Moiseev, Zhuk, 2017]

- There are exactly 1 918 040 clones on 3 elements containing majority.



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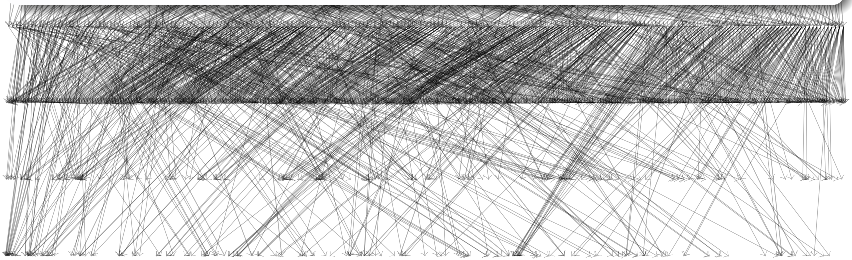

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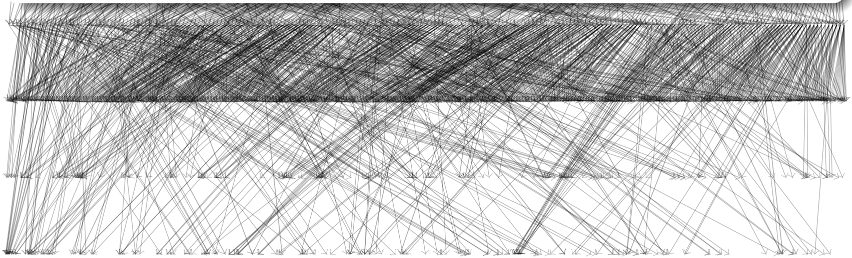

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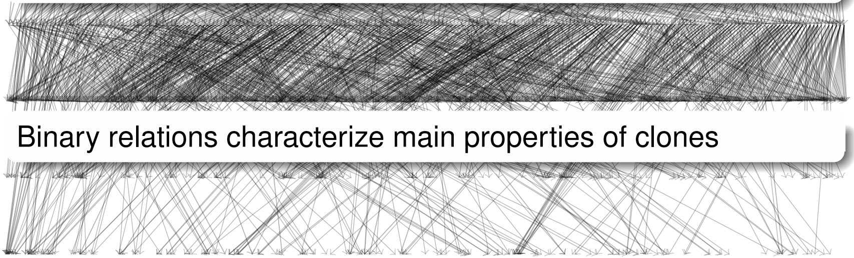

Binary relations characterize main properties of clones

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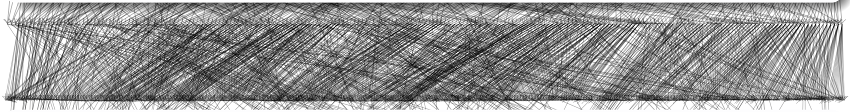

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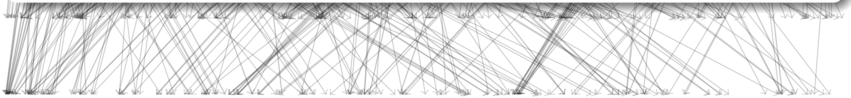

We will never understand that many clones...

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Theorem [Matthew Moore, 2019]

Problem 3 is undecidable.

IV. A New Hope!

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$\mathcal{R}_1$ pp-interpret $\mathcal{R}_2$ if if there exists $d \in \mathbb{N}$ and a partial surjective map $f\colon A_1^d \to A_2$ such that preimages of relations of $\mathcal{R}_2$ are pp definable in $\mathcal{R}_1$.

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Theorem [Birkhoff, Bodirsky]

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- There exists a homomorphism $\xi : \mathcal{C}_1 \to \mathcal{C}_2$
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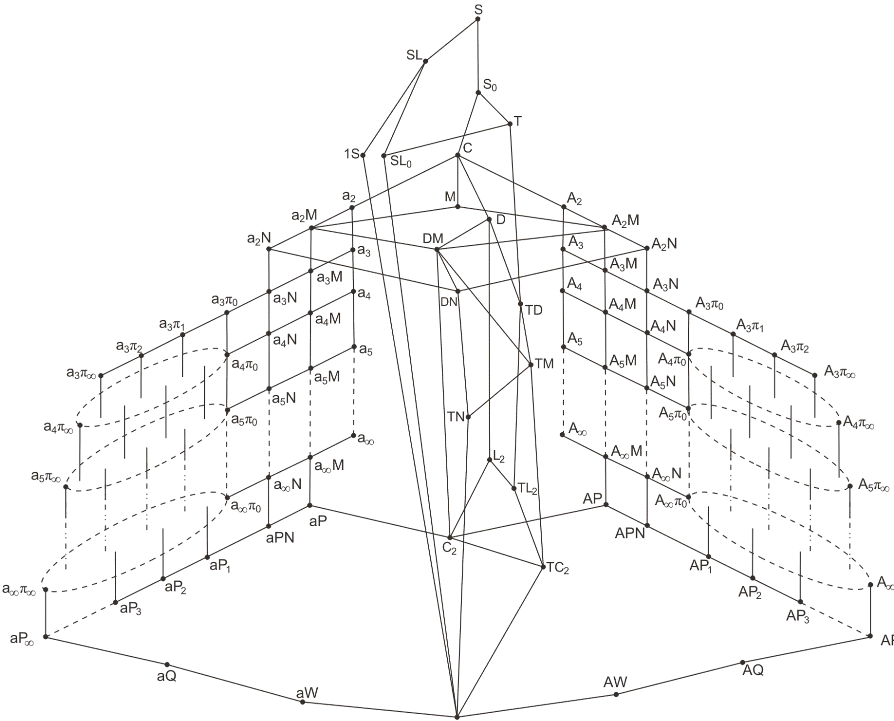
S
SL
S_0
T
1S
SL_0
C
M
a_2
A_2
a_2M
D
A_2M
a_2N
DM
A_2N
a_3
A_3
A_3M
a_3M
$a_3\pi_0$
a_3N
a_4
A_4
DN
TD
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a_4M
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a_5
A_5
$A_3\pi_2$
$a_4\pi_0$
TM
A_5M
$A_4\pi_0$
$a_3\pi_\infty$
a_5M
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a_5N
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TN
$A_5\pi_0$
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a_∞
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L_2
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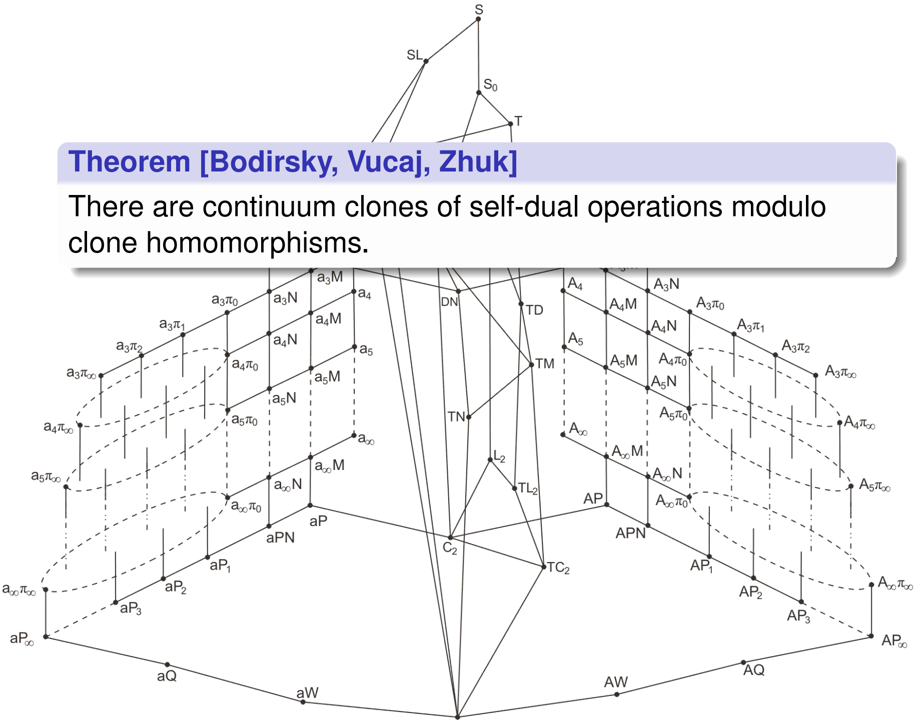
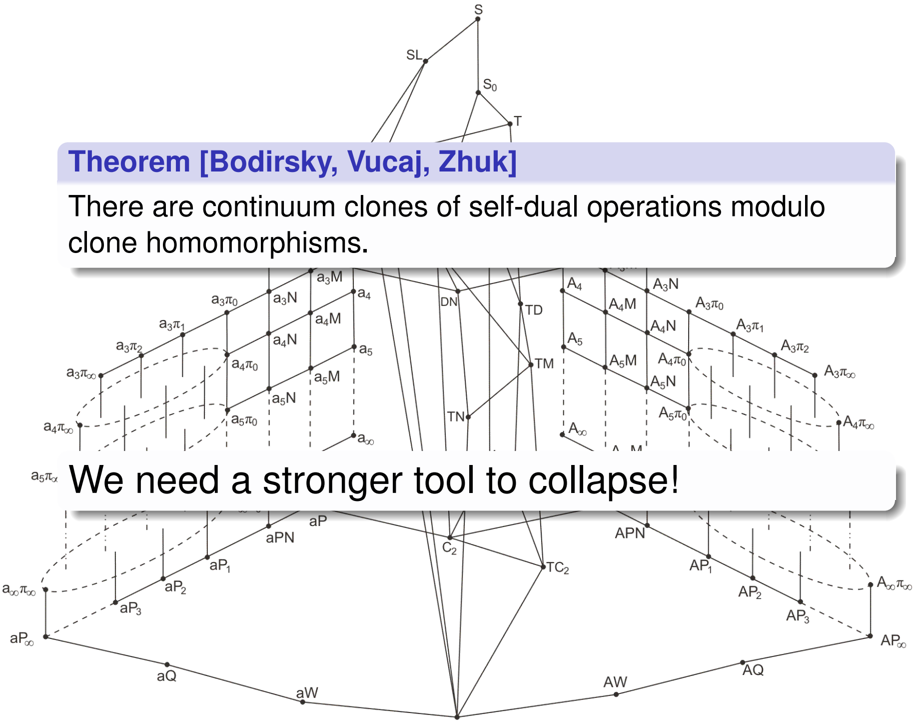
$a_5\pi_\infty$
$a_\infty N$
$A_\infty N$
TL_2
$A_5\pi_\infty$
AP
$A_\infty\pi_0$
$a_\infty\pi_0$
aP
APN
aPN
C_2
aP_1
TC_2
AP_1
aP_2
$a_\infty\pi_\infty$
$A_\infty\pi_\infty$
AP_2
aP_3
AP_3
aP_∞
AP_∞
aQ
AQ
AW
aW



Theorem [Bodirsky, Vucaj, Zhuk]

There are continuum clones of self-dual operations modulo clone homomorphisms.



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We need a stronger tool to collapse!

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Post Lattice

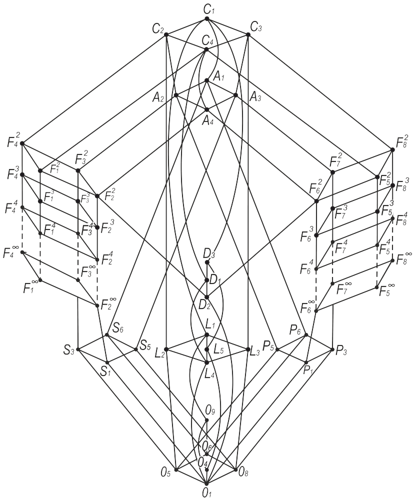

Figure: Post Lattice

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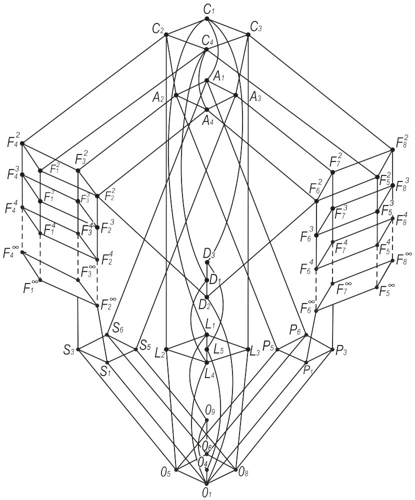

Figure: Post Lattice

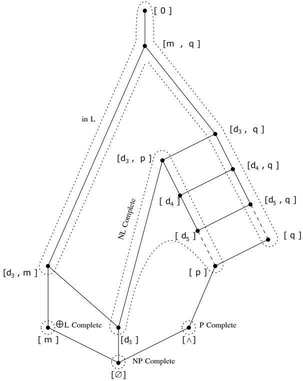

Figure: Post Lattice collapsed

Clones of self-dual operations

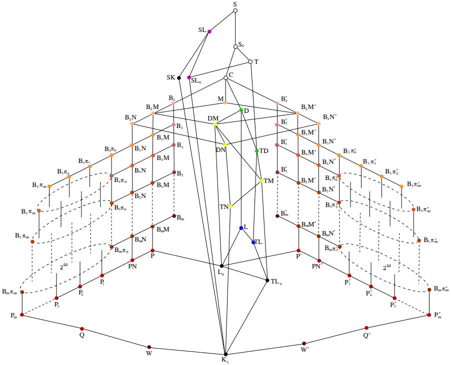

Figure: The lattice of clones of self-dual operations

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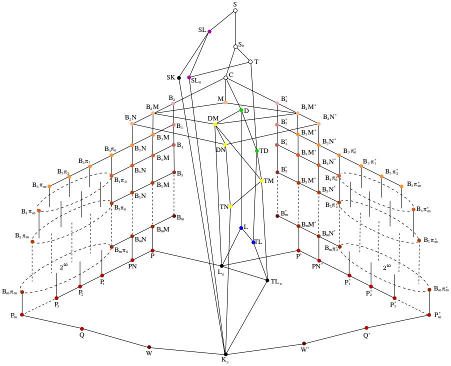

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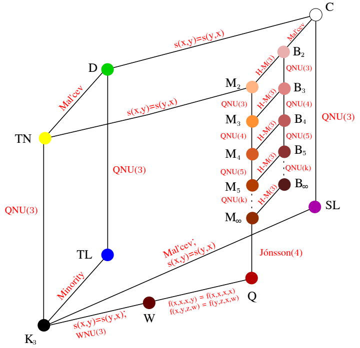

Figure: The lattice of clones of self-dual operations collapsed

V. Continuum Strikes Back

Disappointing family of clones

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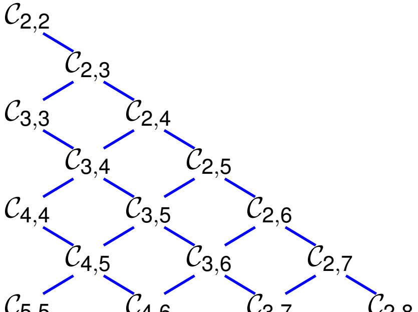
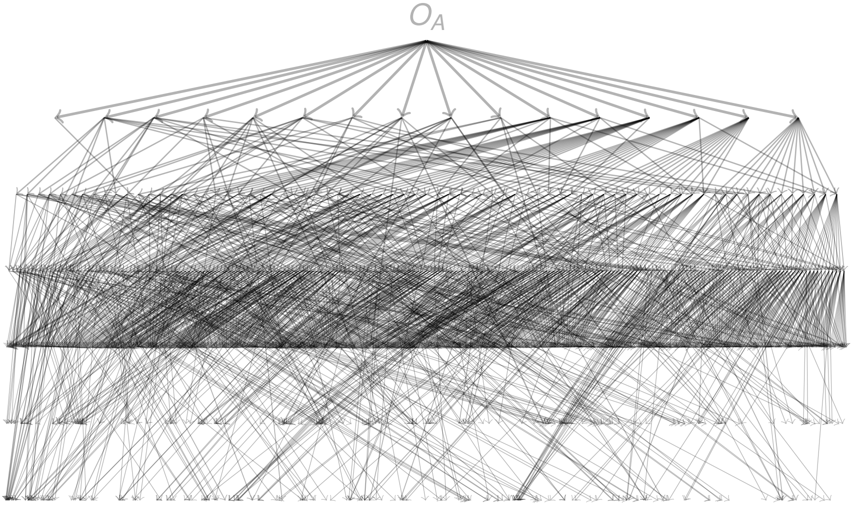


VI. Return of Beautiful Clones


O_A

Computer calculations [Moiseev, Zhuk, 2017]

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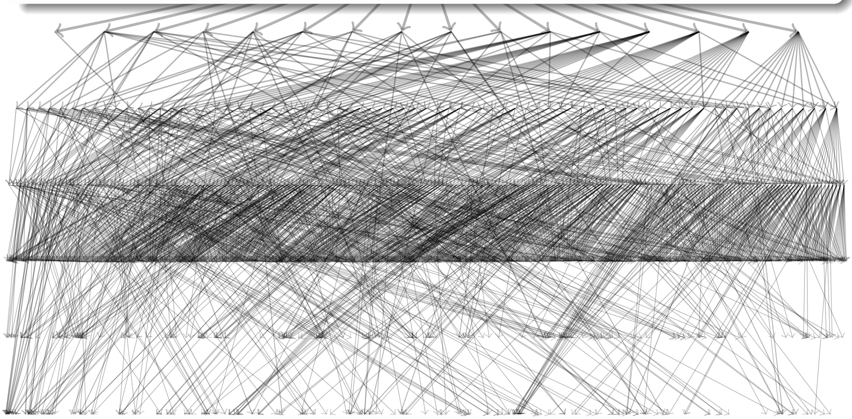

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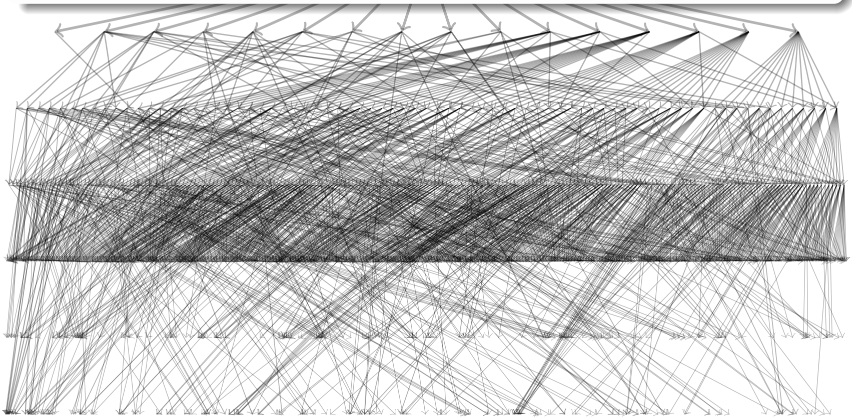

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Collapsing and distinguishing clones by hands.

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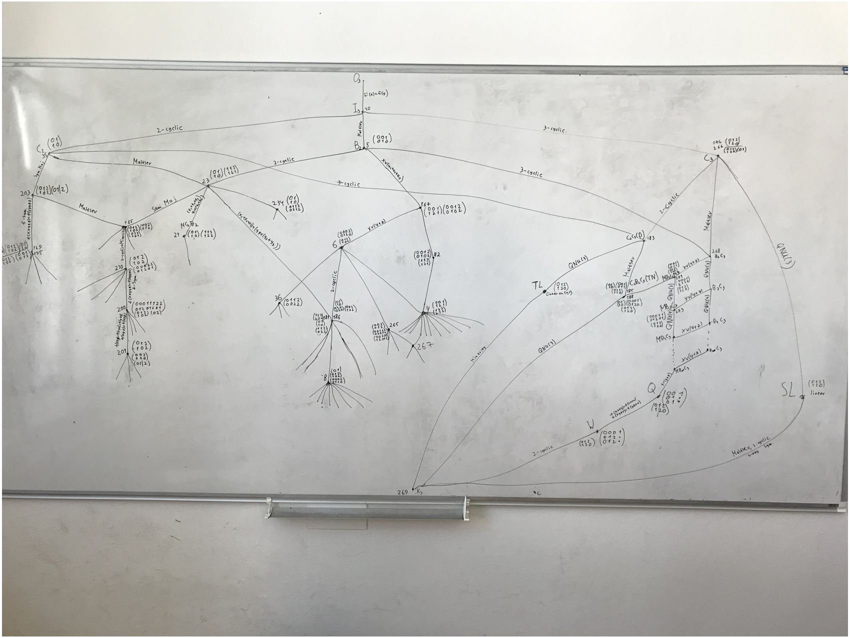

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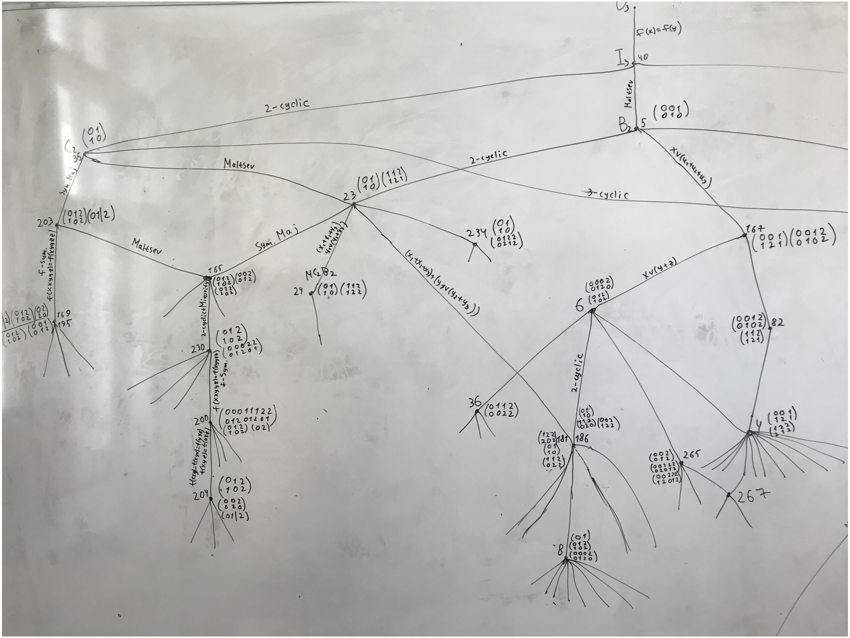

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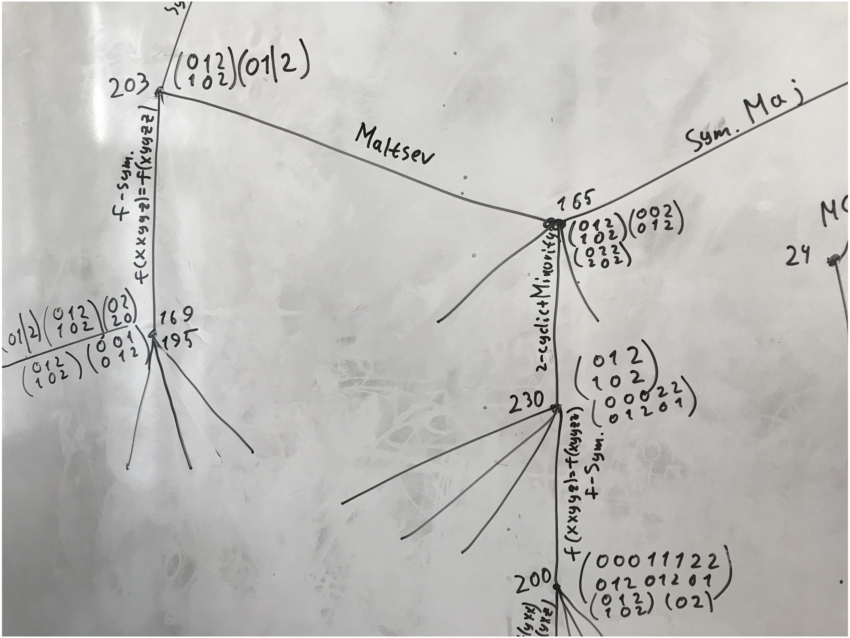
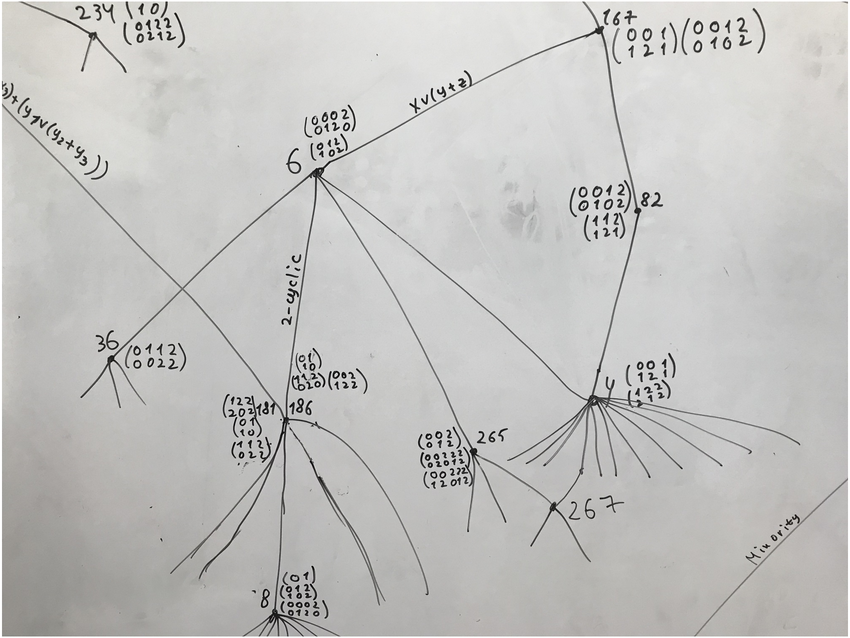


Collapsing and distinguishing clones by hands.



Computer calculations [Moiseev, Zhuk, 2017]

- There are 2 079 040 clones definable by binary relations
- There are 1 656 226 idempotent clones definable by binary relations

Computer Calculations [Zahálka, Barto, Zhuk, Starke, 2022]

- Using second pp-power for $\mathcal{C}_0 \supseteq \mathcal{C}_1$ 1 656 226 were collapsed to 1 297
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Theorem

There are 133+1 clones on 3 elements definable by binary relations up to minor preserving maps.

$\mathcal{O}_3$ = All Operations

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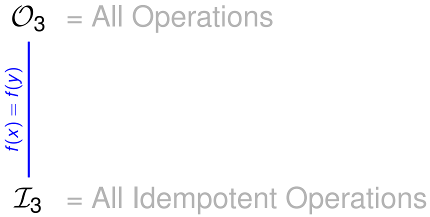
$\mathcal{I}_3$ = All Idempotent Operations

$\mathcal{O}_3$ = All Operations

$$f(x) = f(y)$$

$\mathcal{I}_3$ = All Idempotent Operations



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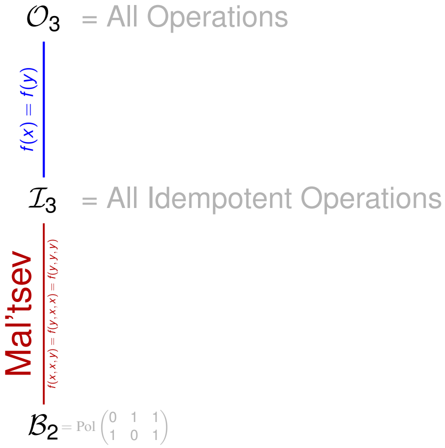

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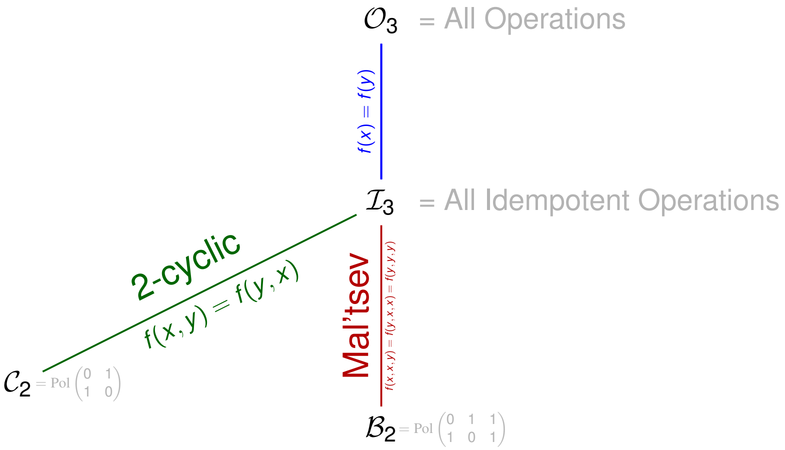
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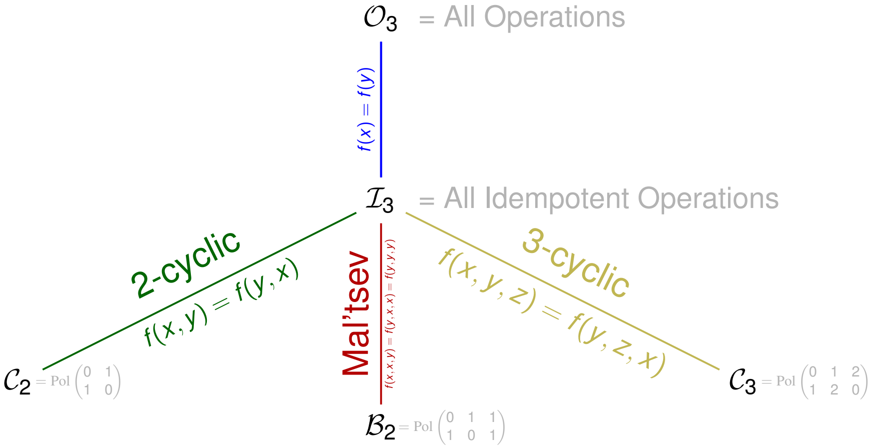

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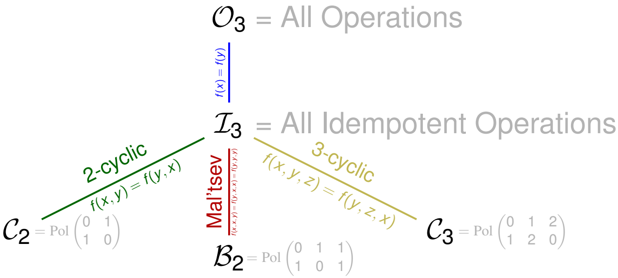

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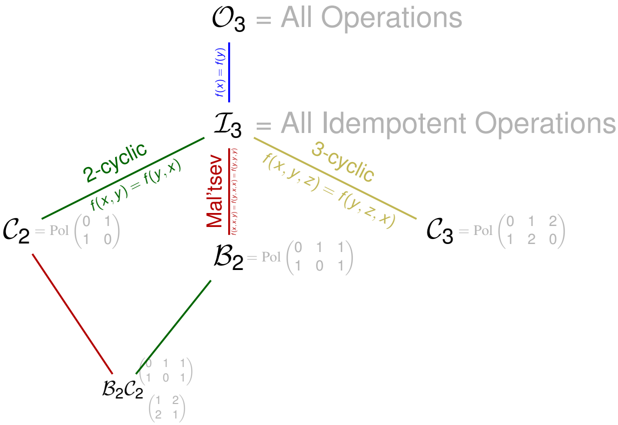

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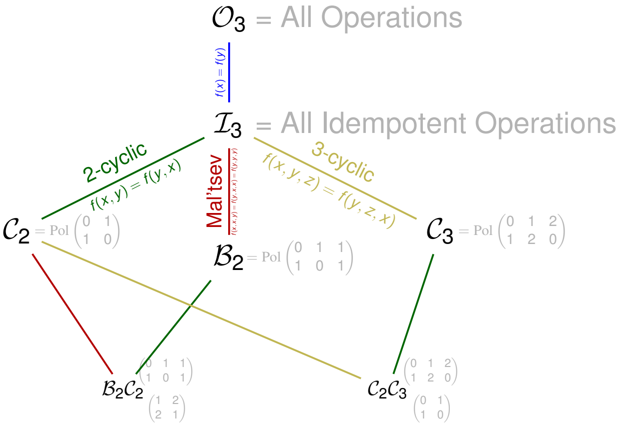

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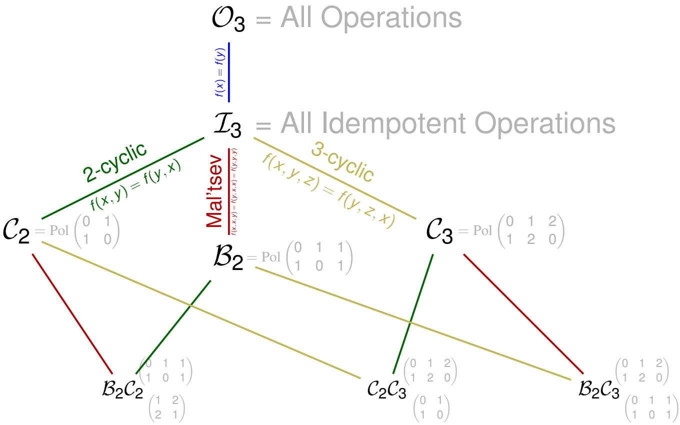

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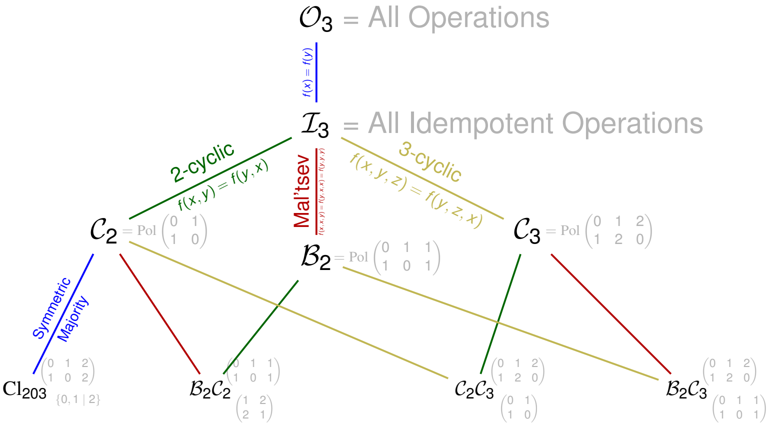
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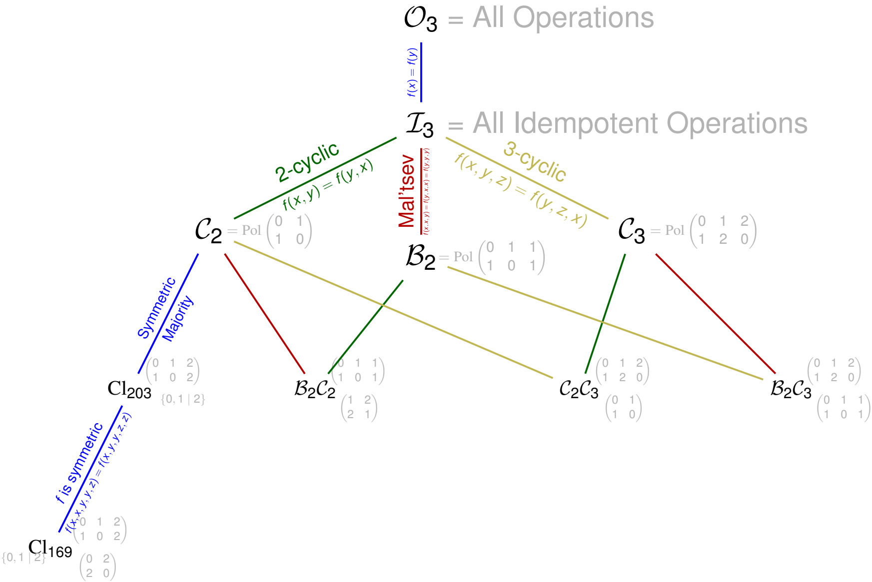

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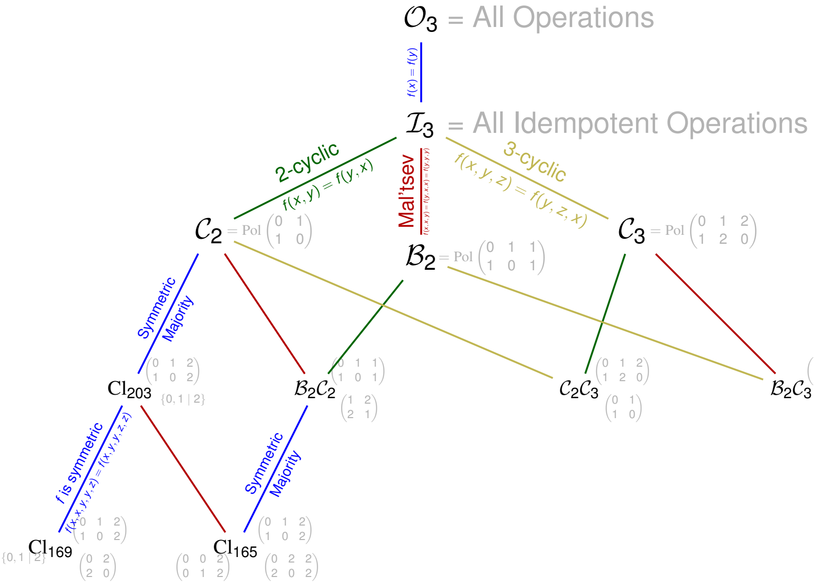

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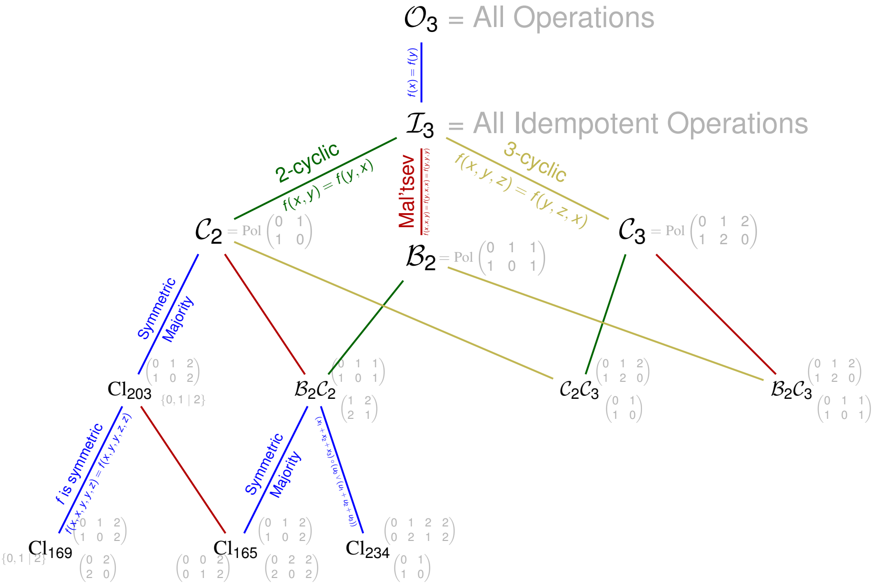

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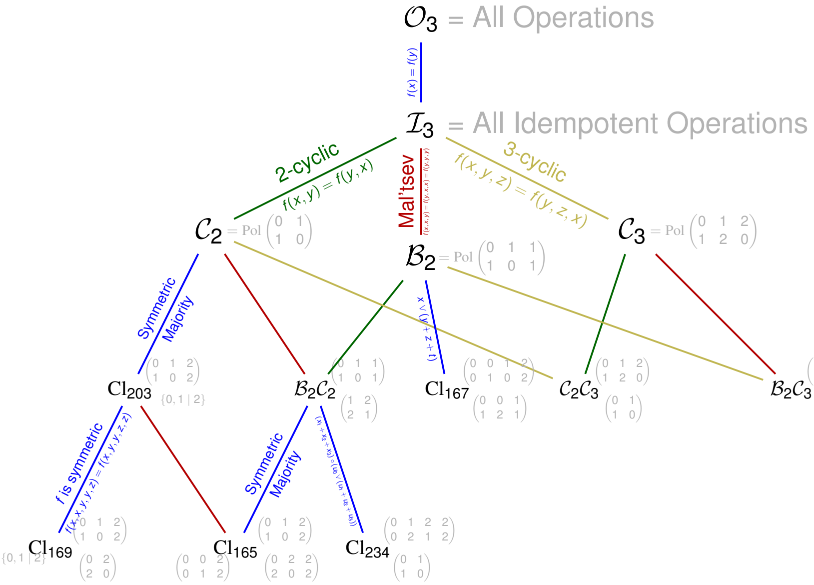

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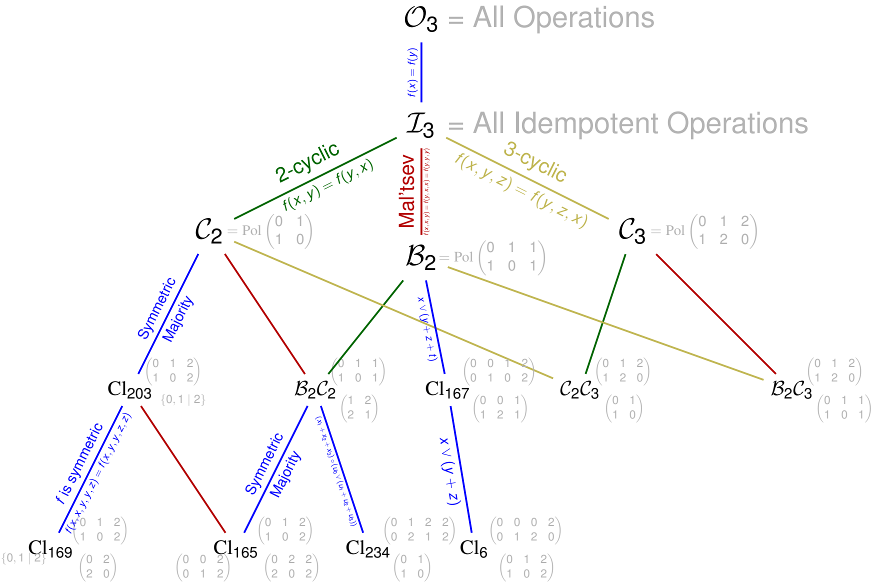

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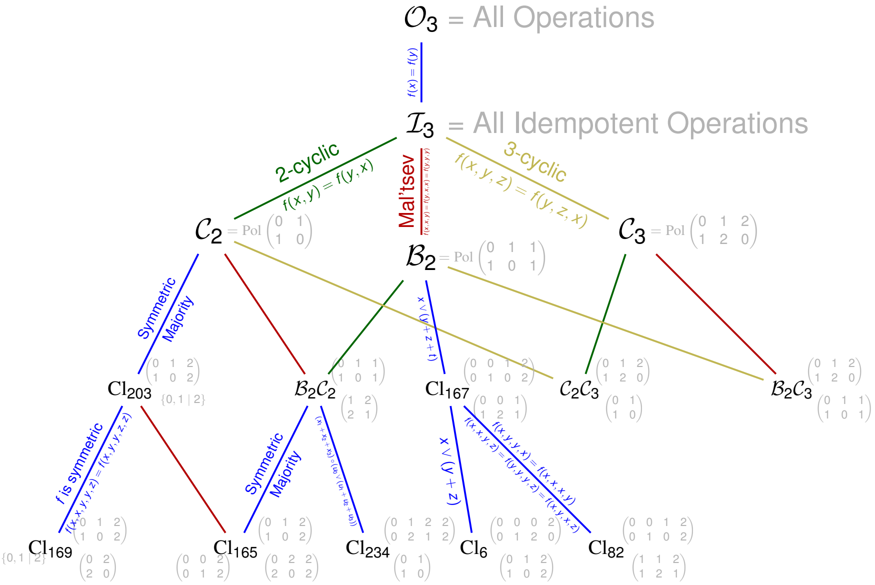

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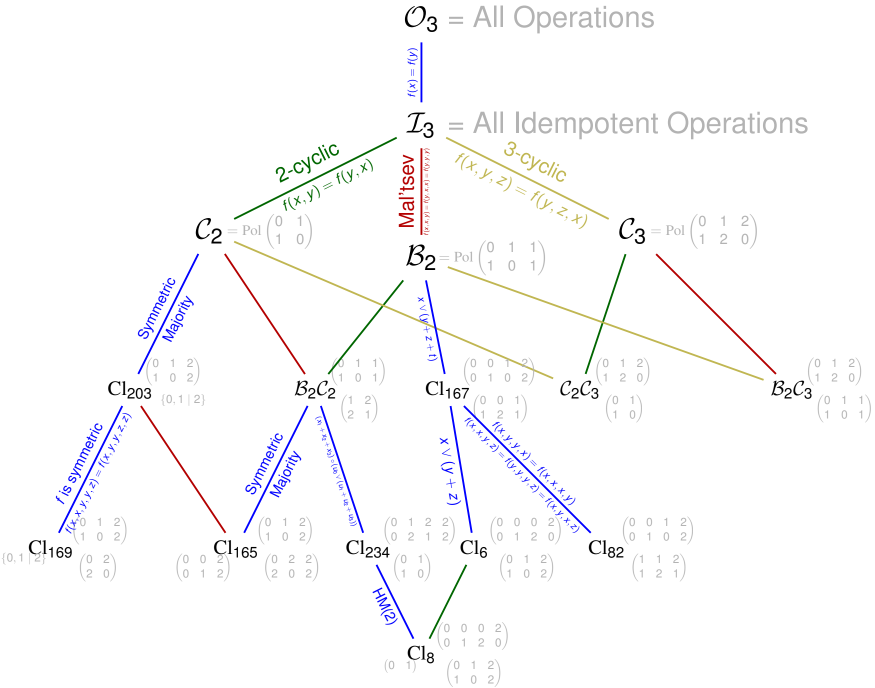

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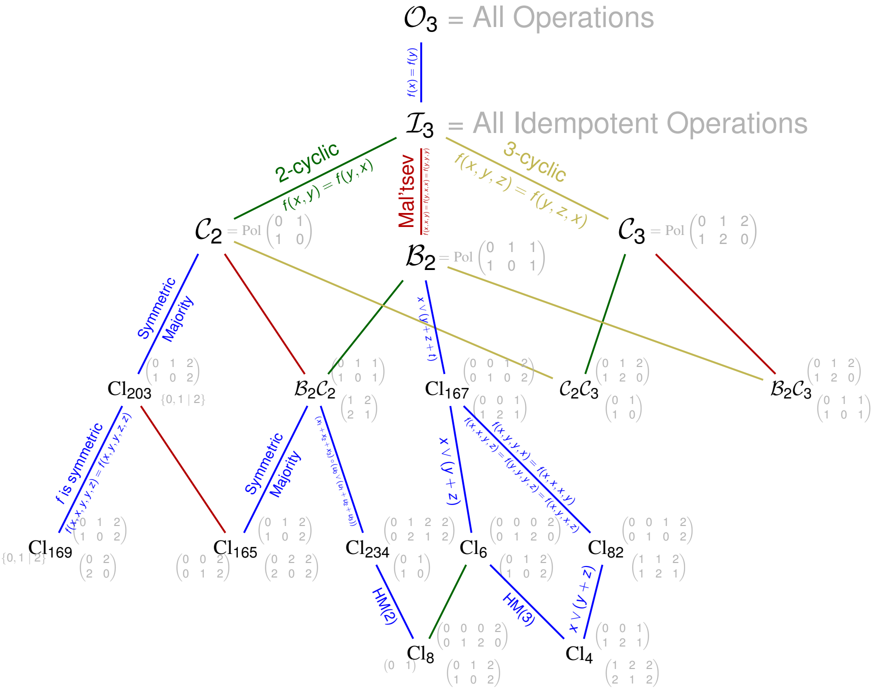

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HM(2)
HM(3)
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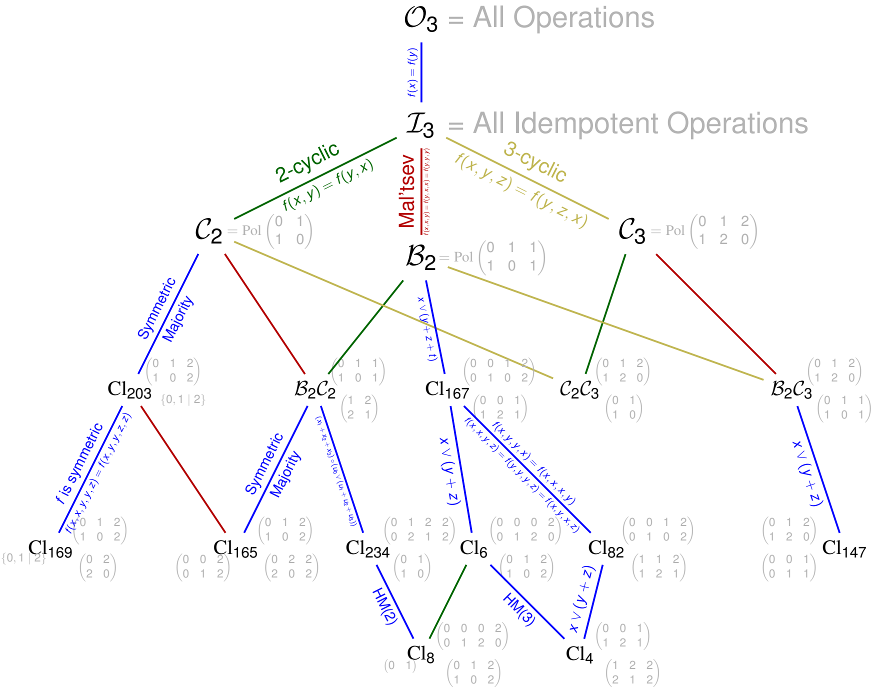
Cl_4



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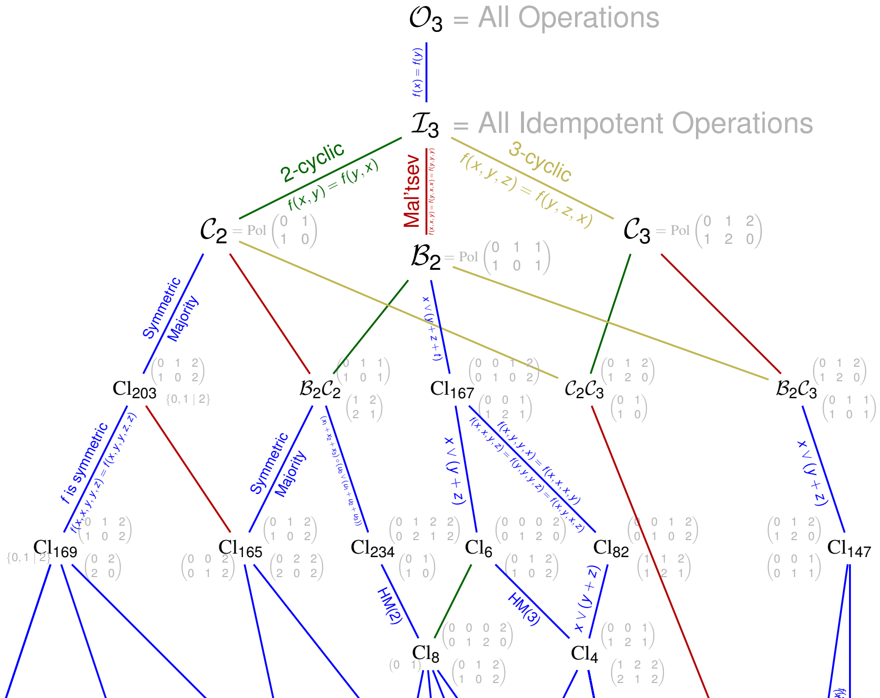
Symmetric Majority
$x \vee (y + z + 1)$
Cl_{203}
$\{0,1 \mid 2\}$
$\mathcal{B}_2\mathcal{C}_2$
Cl_{167}
$\mathcal{C}_2\mathcal{C}_3$
$\mathcal{B}_2\mathcal{C}_3$
f is symmetric
$f(x,x,y,y,z) = f(x,y,y,z,z)$
Symmetric Majority
$x \vee (y + z)$
$f(x,y,y,x) = f(x,x,x,y)$
$x \vee (y + z)$
Cl_{169}
$\{0,1 \mid 2\}$
Cl_{165}
Cl_{234}
Cl_6
Cl_{82}
Cl_{147}
HM(2)
HM(3)
$x \vee (y + z)$
Cl_8
Cl_4



$\mathcal{O}_3$ = All Operations
$f(x) = f(y)$
$\mathcal{I}_3$ = All Idempotent Operations
2-cyclic
$f(x, y) = f(y, x)$
Mal'tsev
3-cyclic
$f(x, y, z) = f(y, z, x)$
$\mathcal{C}_2$ = Pol $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
$\mathcal{B}_2$ = Pol $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$
$\mathcal{C}_3$ = Pol $\begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \end{pmatrix}$
Symmetric
Majority
$x \vee (y + z + 1)$
Cl_{203}
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HM(2)
HM(3)
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Cl_{4}

Interesting part

Interesting part

$\mathit{Domain} = \{0, 1\}$

Interesting part

$Domain = \{0, 1\}$

$\mathcal{M}$

Interesting part

Domain $= \{0, 1\}$

$$\mathcal{M} = \mathrm{Pol}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

Interesting part

Domain $= \{0, 1\}$

$$\mathrm{Clo}(\vee, \wedge) = \mathcal{M} = \mathrm{Pol}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

Interesting part

Domain $= \{0, 1\}$

$$\mathrm{Clo}(\vee, \wedge) = \mathcal{M} = \mathrm{Pol}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \sim \mathcal{M} \cap \mathcal{B}_2 = \mathrm{Pol}\left(\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}\right)$$

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$$\mathrm{Maj} \;= \mathrm{Pol}\left(\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right)$$

Interesting part

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$\mathrm{Clo}(xy \vee xz \vee yz) = \mathrm{Maj} = \mathrm{Pol}\left(\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right)$

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2-cyclic $f(x, y) = f(y, x)$

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Interesting part

$Domain = \{0, 1, 2\}$

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$$\mathrm{Clo}(xy \vee xz \vee yz) = \mathrm{Maj} \;\; = \mathrm{Pol}\left(\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right)$$

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$Domain = \{0, 1, 2\}$

$\mathrm{Clo}(\vee, \wedge) = \mathcal{M} = \mathrm{Pol}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \sim$ 1 329 769 of clones

$\mathrm{Clo}(xy \vee xz \vee yz) = \mathrm{Maj} = \mathrm{Pol}\left(\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right)$

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$\mathrm{Clo}(xy \vee xz \vee yz) = \mathrm{Maj} \quad = \mathrm{Pol}\left(\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right) \sim$ 93 840 of clones

Interesting part

$$\textit{Domain} = \{0, 1, 2\}$$

$$\mathrm{Clo}(\vee, \wedge) = \mathcal{M} = \mathrm{Pol}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \sim \text{1 329 769 of clones}$$

$$\mathrm{CL}_3 = \mathrm{Pol}\left(\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\right)$$

$$\mathrm{CL}_{19} = \mathrm{Pol}\left(\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 1 & 2 & 2 \end{pmatrix}\right)$$

$$\mathrm{CL}_{46} = \mathrm{Pol}\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 1 & 2 \\ 0 & 1 & 2 & 2 & 2 \end{pmatrix}\right)$$

$$\mathrm{Clo}(xy \vee xz \vee yz) = \mathrm{Maj} = \mathrm{Pol}\left(\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right) \sim \text{93 840 of clones}$$

Interesting part

Domain $= \{0, 1, 2\}$

$\mathrm{Clo}(\vee, \wedge) = \mathcal{M} = \mathrm{Pol}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ ~ 1 329 769 of clones

2-cyclic $f(x, y) = f(y, x)$

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f is symmetric

$f(x, x, x, y, z) = f(x, y, z, z, z)$

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2-cyclic: $f(x, y) = f(y, x)$

$\mathrm{CL}_3 = \mathrm{Pol}\left(\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\right)$

2-cyclic + minority:

$f(x_1, x_2, u_1, u_2, u_3) = f(x_2, x_1, u_1, u_2, u_3)$

$f(x_1, x_2, u_1, u_2, u_3) = f(x_1, x_2, u_2, u_1, u_3) = f(x_1, x_2, u_2, u_3, u_1)$

$f(x_1, x_2, u, u', u') = f(x_1, x_2, u, u, u)$

$\mathrm{CL}_{19} = \mathrm{Pol}\left(\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 1 & 2 & 2 \end{pmatrix}\right)$

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$f(x, x, z, z, y) = f(y, x, y, x, x)$

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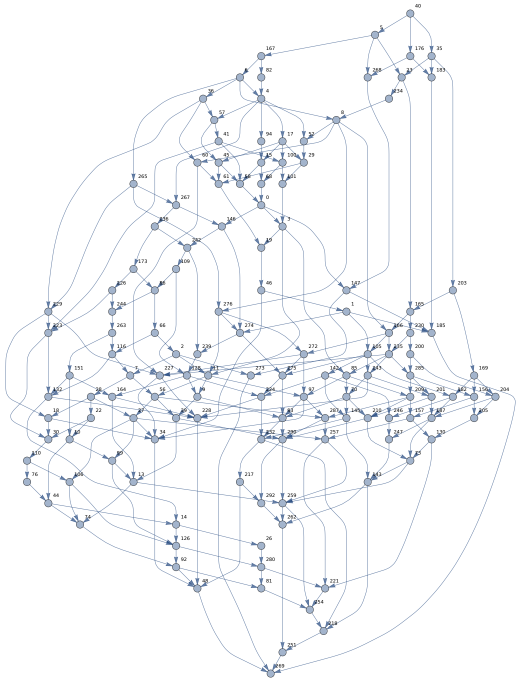
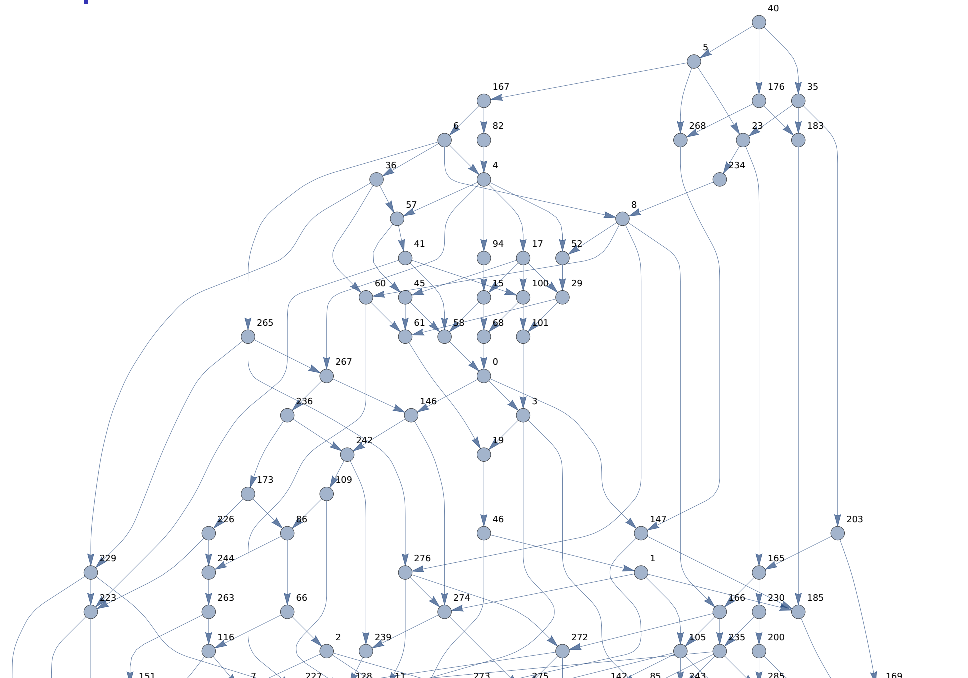
Full picture



Full picture



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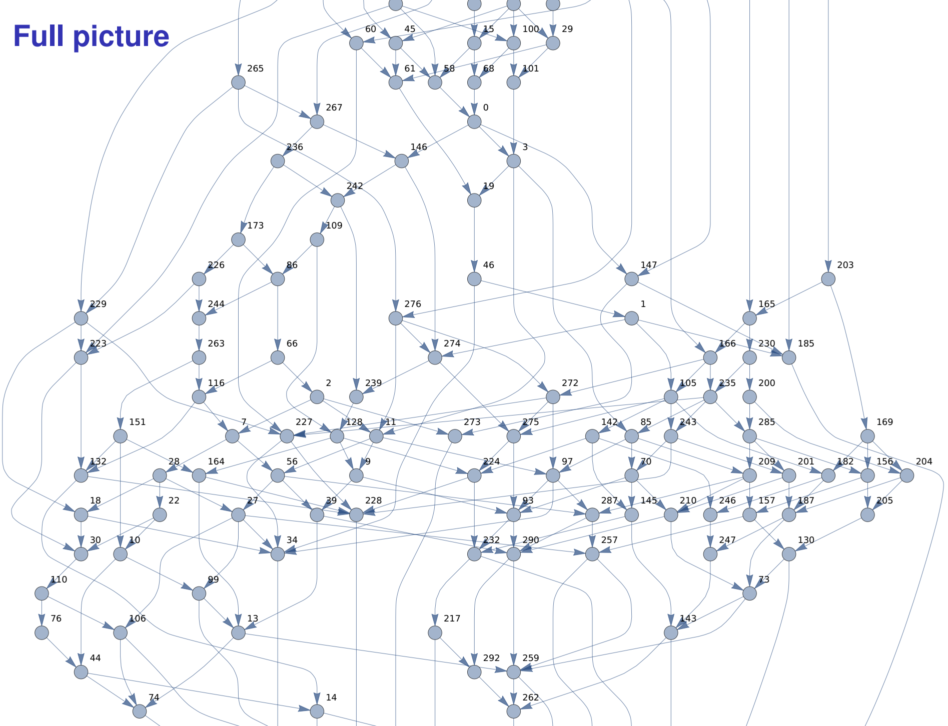

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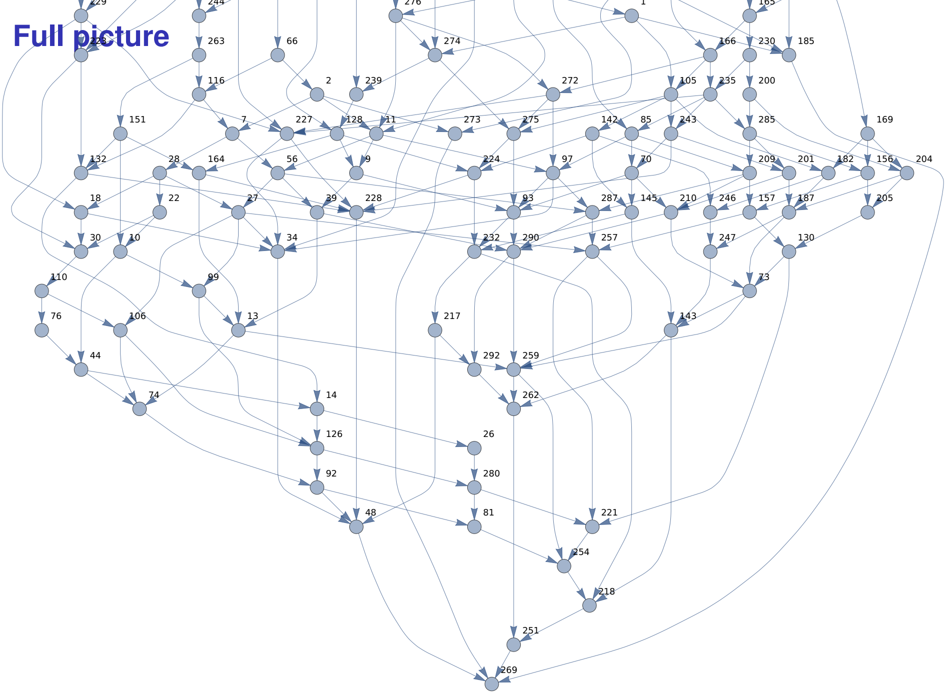

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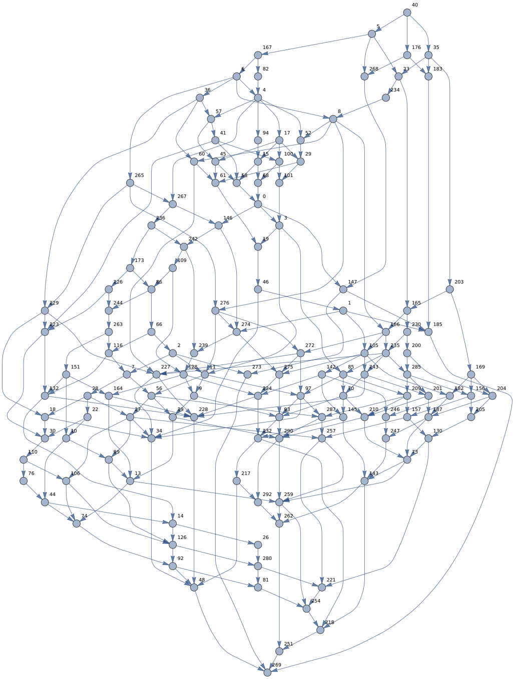

This is not a lattice

This is not a lattice

Claim

Clones on 3 elements modulo minor preserving maps do not form a lattice.

This is not a lattice

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Clones on 3 elements modulo minor preserving maps do not form a lattice.

$$\operatorname{Pol}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

$$\mathcal{M}$$

This is not a lattice

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Clones on 3 elements modulo minor preserving maps do not form a lattice.

$$\underset{\mathcal{M}}{\mathrm{Pol}\begin{pmatrix}0 & 0 & 1\\ 0 & 1 & 1\end{pmatrix}} \qquad \underset{\mathrm{CL}_{203}}{\mathrm{Pol}\left(\begin{pmatrix}0 & 1 & 2\\ 1 & 0 & 2\end{pmatrix}, \{0,1 \mid 2\}\right)}$$

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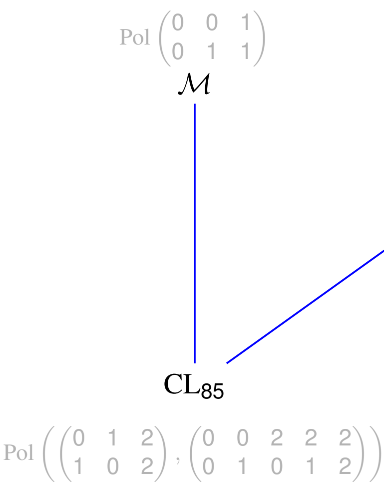

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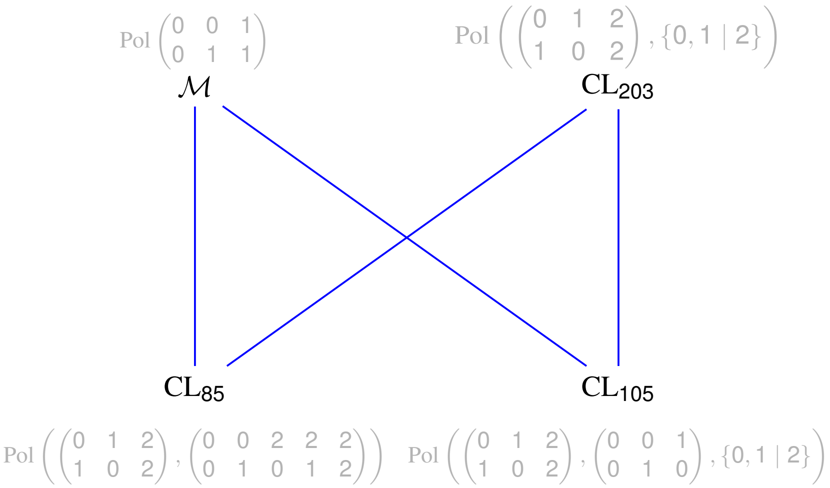

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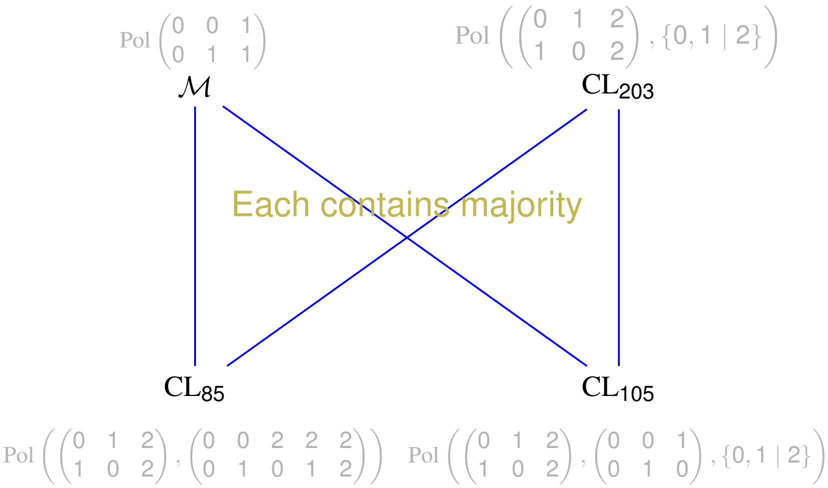

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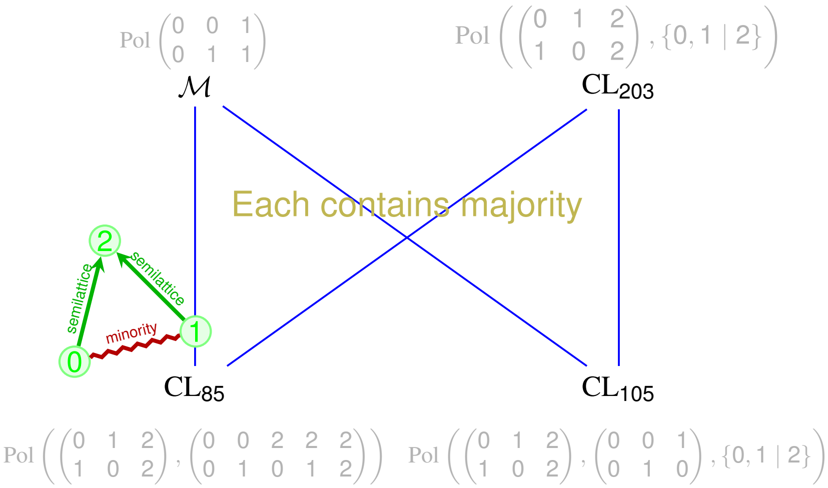

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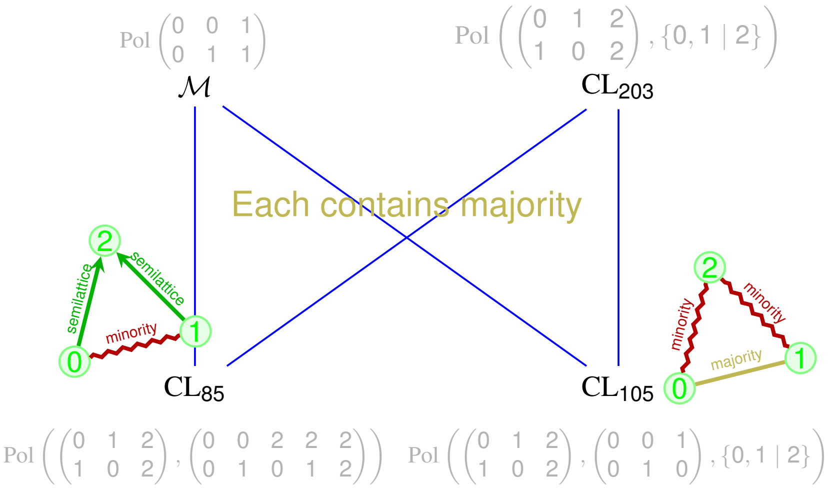

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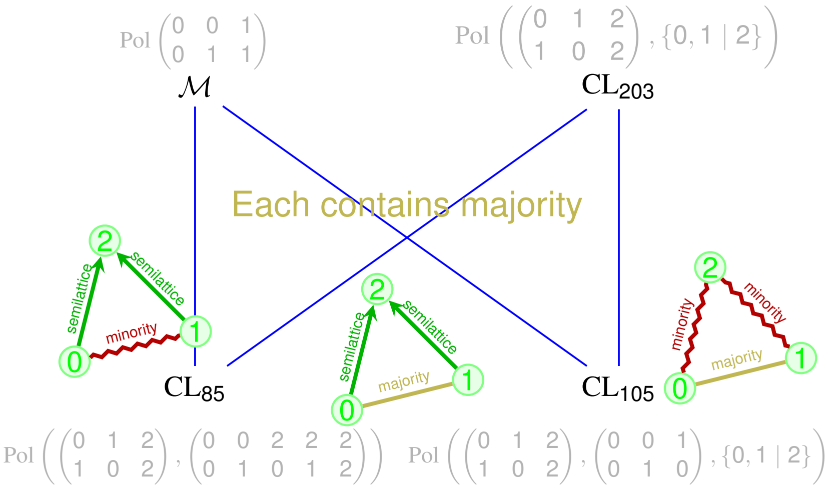

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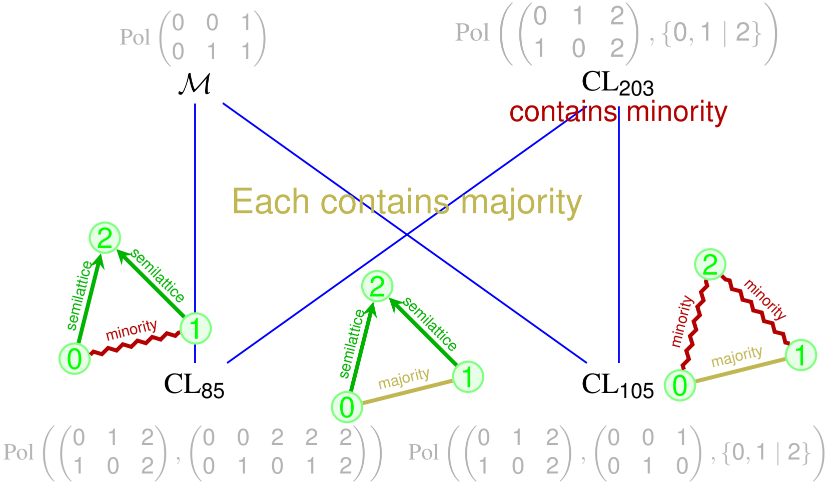

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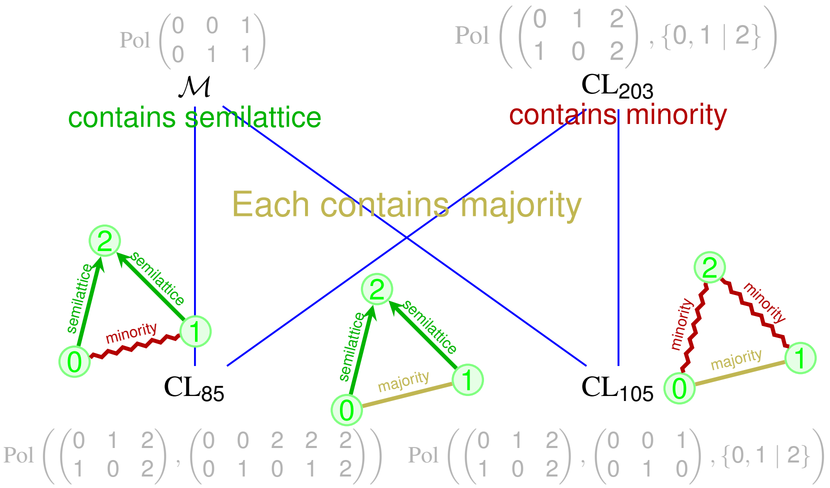
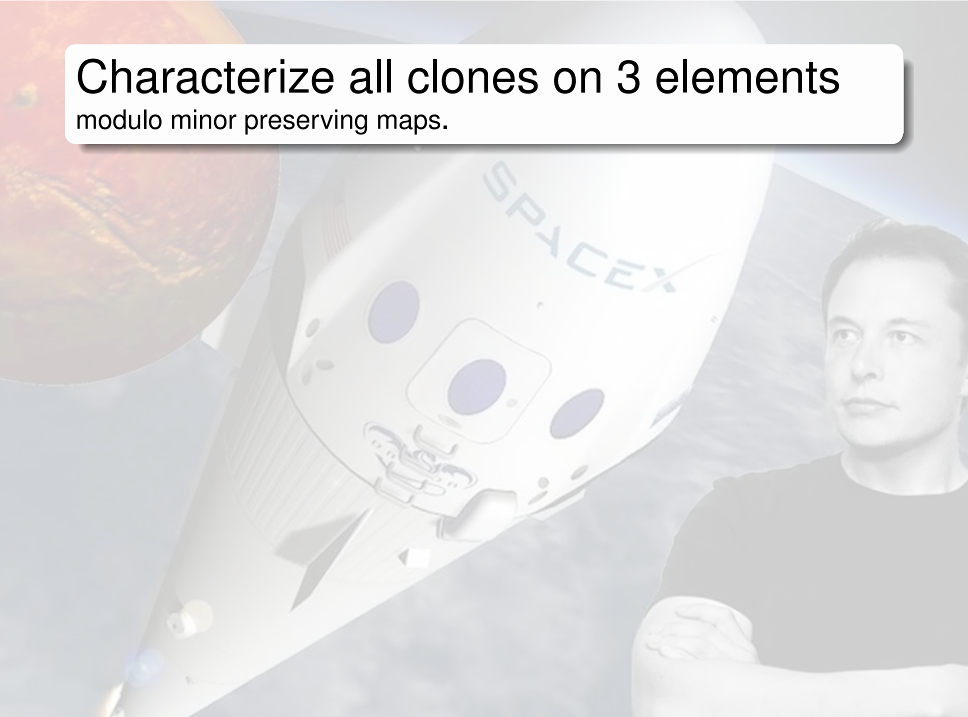


Characterize all clones on 3 elements

modulo minor preserving maps.



Characterize all clones on 3 elements

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Plan

Characterize all clones on 3 elements

modulo minor preserving maps.

Plan

1. Classify all clones of self-dual operations modulo minor preserving maps

Characterize all clones on 3 elements

modulo minor preserving maps.

Plan

1. Classify all clones of self-dual operations modulo minor preserving maps ✓

Characterize all clones on 3 elements

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Plan

1. Classify all clones of self-dual operations modulo minor preserving maps ✓
2. Take all minimal Taylor clones and characterize them modulo minor preserving maps

Characterize all clones on 3 elements

modulo minor preserving maps.

Plan

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Characterize all clones on 3 elements
modulo minor preserving maps.

Plan

1. Classify all clones of self-dual operations modulo minor preserving maps ✓
2. Take all minimal Taylor clones and characterize them modulo minor preserving maps ✓
3. Take 1 656 226 clones definable by binary relations and characterize them modulo minor preserving maps

Characterize all clones on 3 elements

modulo minor preserving maps.

Plan

1. Classify all clones of self-dual operations modulo minor preserving maps ✓
2. Take all minimal Taylor clones and characterize them modulo minor preserving maps ✓
3. Take 1 656 226 clones definable by binary relations and characterize them modulo minor preserving maps ✓

Characterize all clones on 3 elements

modulo minor preserving maps.

Plan

1. Classify all clones of self-dual operations modulo minor preserving maps ✓
2. Take all minimal Taylor clones and characterize them modulo minor preserving maps ✓
3. Take 1 656 226 clones definable by binary relations and characterize them modulo minor preserving maps ✓
4. Find all submaximal elements in the poset

Characterize all clones on 3 elements

modulo minor preserving maps.

Plan

1. Classify all clones of self-dual operations modulo minor preserving maps ✓
2. Take all minimal Taylor clones and characterize them modulo minor preserving maps ✓
3. Take 1 656 226 clones definable by binary relations and characterize them modulo minor preserving maps ✓
4. Find all submaximal elements in the poset ✓

Characterize all clones on 3 elements

modulo minor preserving maps.

Plan

1. Classify all clones of self-dual operations modulo minor preserving maps ✓
2. Take all minimal Taylor clones and characterize them modulo minor preserving maps ✓
3. Take 1 656 226 clones definable by binary relations and characterize them modulo minor preserving maps ✓
4. Find all submaximal elements in the poset ✓
5. Find all subsubmaximal elements in the poset

Characterize all clones on 3 elements

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Plan

1. Classify all clones of self-dual operations modulo minor preserving maps ✓
2. Take all minimal Taylor clones and characterize them modulo minor preserving maps ✓
3. Take 1 656 226 clones definable by binary relations and characterize them modulo minor preserving maps ✓
4. Find all submaximal elements in the poset ✓
5. Find all subsubmaximal elements in the poset
6. Classify all Mal'tsev Clones modulo minor preserving maps

Characterize all clones on 3 elements

modulo minor preserving maps.

Plan

1. Classify all clones of self-dual operations modulo minor preserving maps ✓
2. Take all minimal Taylor clones and characterize them modulo minor preserving maps ✓
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4. Find all submaximal elements in the poset ✓
5. Find all subsubmaximal elements in the poset
6. Classify all Mal'tsev Clones modulo minor preserving maps
7. ...

Let us dream...

Let us dream...

What will a full description of all clones modular minors give us?

Let us dream...

What will a full description of all clones modular minors give us?

- Beautiful picture?

Let us dream...

What will a full description of all clones modular minors give us?

- Beautiful picture? Hopefully

Let us dream...

What will a full description of all clones modular minors give us?

- Beautiful picture? Hopefully
- A handbook of clones and h1-identities on 3 elements

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  M. B.: Can we generate majority from 3-cyclic and Maltsev?

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Thank you for your attention!