Valued Constraint Satisfaction Problem and Resilience in Database Theory

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Outline

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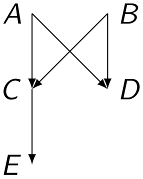
Queries and databases

Database: a relational structure $\mathfrak{A}$

parent	of
Alice	Cecilia
Bob	Cecilia
Alice	Daniel
Bob	Daniel
Cecilia	Eva



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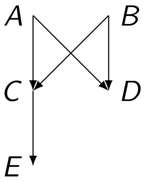

Conjunctive query: a primitive positive formula q, i. e. a formula of the form

$$\exists y_1, \ldots, y_l(\psi_1 \wedge \cdots \wedge \psi_m),$$

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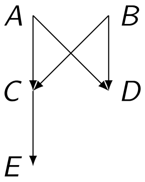

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Example: $\mathfrak{A}$ as above, $q := \exists x, y, z(\text{parent}(x, y) \wedge \text{parent}(y, z))$, then $\mathfrak{A} \models q$ with $x = A$, $y = C$ and $z = E$.

Resilience

Definition (resilience)

Fixed conjunctive query q. Problem RES(q):
Input: a finite database $\mathfrak{A}$
Output: minimum number of tuples to be removed from relations of $\mathfrak{A}$ so that $\mathfrak{A} \not\models q$

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Example: The resilience of $\mathfrak{A}$ with respect to

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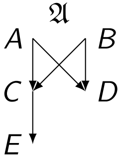
is 1 – remove either (A, C) or (C, E).



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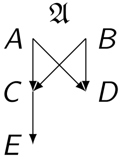

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Research goal: Classify complexity of resilience for all conjunctive queries.

Variants of resilience

Two variants of databases:

- set semantics: each tuple occurs at most once
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For a conjunctive query q:

- RES(q) tractable in bag semantics
  $\Rightarrow$ RES(q) tractable in set semantics
- RES(q) NP-hard in set semantics
  $\Rightarrow$ RES(q) NP-hard in bag semantics

Examples

- $q_{\text{path}} := \exists x, y, z\big(R(x, y) \wedge S(y, z)\big)$
- $q_{\triangle} := \exists x, y, z\big(R(x, y) \wedge S(y, z) \wedge T(z, x)\big)$
- $q'_{\triangle} := \exists x, y\big(A(x) \wedge R(x, y) \wedge S(y, z) \wedge T(z, x) \wedge B(z)\big)$
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query	**set semantics**	**bag semantics**
q_{path}	P (MGMS)	P (BLS)
$q_{\triangle}$	NP-hard (FGIM)	NP-hard (FGIM)
$q'_{\triangle}$	P (FGIM)	NP-hard (MG)
q_{new}	P (BLS)	P (BLS)

References:

- Meliou, Gatterbauer, Moore, Suciu ('10)
- Freire, Gatterbauer, Immerman, Meliou ('15)
- Makhija, Gatterbauer ('22)
- Bodirsky, Lutz, S.

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Constraint satisfaction

Fixed τ-structure $\mathfrak{A}$ (τ – finite relational signature)
Input: list of atomic τ-formulas (constraints)
Output:

- **CSP**: Decide whether there is a solution that satisfies all constraints.
- **MaxCSP**: Find the maximal number of constraints that can be satisfied at once.
- **VCSP**: Find the minimal cost with which the constraints can be satisfied (each constraint comes with a cost depending on the chosen values).

Might be considered with a threshold.

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Observation: VCSP generalizes CSP and MaxCSP.
Proof: Model the tuples in relations with cost 0 and outside with cost

- 1 and the same threshold (for MaxCSP);
- ∞ and threshold 0 (for CSP).

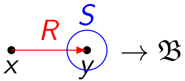
Different formulations of constraint satisfaction

Satisfying a list of constraints can be viewed alternatively as:

- satisfying a primitive positive formula
- being able to map the corresponding structure homomorphically
- a sum of the constraints (in the valued setting) being below a threshold

Example: Consider the list $R(x, y)$, $S(y)$. A structure $\mathfrak{B}$ satisfies these constraints iff:

- $\mathfrak{B} \models \exists x, y(R(x, y) \wedge S(y))$
- the canonical structure maps homomorphically to $\mathfrak{B}$, i.e.,



- the sum $R(x, y) + S(y)$ in $\mathfrak{B}$ is 0

Focus on VCSP

A valued structure Γ, consists of:

- (countable) domain C
- (finite, relational) signature τ
- for each $R \in \tau$ of arity k, a function $R^\Gamma \colon C^k \to \mathbb{Q} \cup \{\infty\}$

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Definition (VCSP(Γ))

Input: $u \in \mathbb{Q}$, an expression

$$\phi(x_1, \ldots, x_n) = \sum_i \psi_i,$$

where each ψ_i is an atomic τ-formula

Question: Is

$$\inf_{\bar{a} \in C^n} \phi(\bar{a}) \leq u \text{ in } \Gamma?$$

Max-Cut as a VCSP

Example:
Input: $G = (V, E)$ – finite directed graph
Goal: Find a partition $A \cup B$ of V such that $E \cap (A \times B)$ is maximal.
Equivalently: $E \cap (A^2 \cup B^2 \cup B \times A)$ is minimal.

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Let Γ_{MC} be a valued structure where:

- $C = \{0, 1\}$
- $\tau = \{E\}$, E binary

$$E(x, y) = \begin{cases} 0 \text{ if } x = 0 \text{ and } y = 1 \\ 1 \text{ otherwise} \end{cases}$$

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Take vertices of G as variables. The size of a maximal cut of G is

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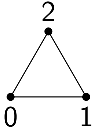
every instance of VCSP(Γ_{MC}) corresponds to a digraph
$\rightsquigarrow$ VCSP(Γ_{MC}) is the Max-Cut problem (NP-hard)

Graph colorability as a VCSP

K_3 is the valued structure on $\{0, 1, 2\}$ with single binary relation E defined:

$$E(x, y) = \begin{cases} 0 \text{ if } x \neq y \\ \infty \text{ if } x = y \end{cases}$$



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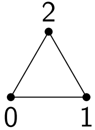

Observation: $\text{VCSP}(K_3) = \text{CSP}(K_3)$ is the 3-colorability problem and hence NP-hard.
More generally, $\text{VCSP}(K_n)$ is the n-colorability problem.

Dichotomy for finite-domain VCSPs

Theorem

If the domain of Γ *is finite,* VCSP(Γ) *is in P or NP-complete.*

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  If the hardness condition does not apply, Γ has a cyclic fractional polymorphism of arity at least two.

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- Bulatov ('17); Zhuk ('17): Proof of Feder-Vardi conjecture.

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Homomorphism duality

For a query q, take its canonical structure $\mathfrak{Q}$.
Search for a structure $\mathfrak{B}_q$ such that for every finite $\mathfrak{A}$:

$$\mathfrak{A} \not\models q \;\Leftrightarrow\; \mathfrak{Q} \not\to \mathfrak{A} \;\Leftrightarrow\; \mathfrak{A} \to \mathfrak{B}_q$$

$\rightsquigarrow$ corresponds to the CSP($\mathfrak{B}_q$)
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Recall the valued structure Γ_{MC}: it is a valued version of the structure P_1.
Observation: $\mathrm{VCSP}(\Gamma_{\mathrm{MC}})$, i.e., the Max-Cut problem is the same problem as the resilience of

$$\exists x, y (E(x, y) \wedge E(y, z)).$$

In particular, it is NP-hard.

Existence of finite dual structures

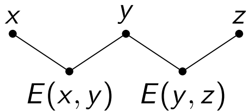
Definition (incidence graph)

The incidence graph $I(q)$ of a query q is an undirected bipartite graph where:

- the first class contains variables of q,
- the second class contains conjuncts of q,
- edges link conjuncts with their variables.

Example: $q := \exists x, y, z(E(x, y) \wedge E(y, z))$

$I(q)$:



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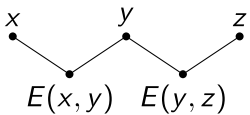

Theorem (Nešetřil,Tardiff ('00); Larose, Loten, Tardiff ('07))

A conjunctive query q has a finite dual if and only if $I(q)$ is a tree.

Complexity with finite duals

Corollary (Bodirsky, Lutz, S.)

Let q be a conjunctive query such that $I(q)$ is a tree. Then the resilience problem for q in bag semantics is in P or NP-complete.

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Proof idea:

- Obtain the finite dual structure $\mathfrak{B}_q$.
- Turn it into a valued structure Γ_q with cost functions taking values 0 and 1.
- RES(q) is the same problem as VCSP(Γ_q) if considering bag databases.
- VCSP(Γ_q) is in P or NP-complete by the dichotomy theorem for finite-domain VCSPs.

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Question: What about general queries? Is there such a structure $\mathfrak{B}_q$?

Infinite dual structures

Theorem (Cherlin, Shelah, Shi ('99))

If I(q) is connected, then q has a countable dual $\mathfrak{B}_q$. $\mathfrak{B}_q$ can be chosen so that $\mathrm{Aut}(\mathfrak{B}_q)$ is oligomorphic.

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oligomorphic – countable domain B_q and the action of $\mathrm{Aut}(\mathfrak{B}_q)$ on B_q^n has finitely many orbits for every $n \geq 1$

Example: $\mathrm{Aut}(\mathbb{Q}; <)$ is oligomorphic.
(However, $(\mathbb{Q}; <)$ is not a dual of a single conjunctive query.)

Connection of resilience and VCSPs

query q with $I(q)$ connected $\rightsquigarrow$ obtain the dual structure $\mathfrak{B}_q$ $\rightsquigarrow$ turn it into a valued structure Γ_q with cost functions taking values 0 and 1

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- presence of exogenous tuples – specified tuples may not be removed $\rightsquigarrow$ use cost ∞ instead of 1 in Γ_q
- holds for finite disjunctions of conjunctive queries
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Remark: Since Aut(Γ_q) is oligomorphic, all relations attain only finitely many different values.

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Expressive power of valued structures

Γ – valued τ-structure with countable domain C, Aut(Γ) oligomorphic
$R\colon C^k \to \mathbb{Q} \cup \{\infty\}$
τ-expression: $\sum_i \psi_i$, ψ_i atomic τ-formulas

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Definition

R is expressible in Γ if for some τ-expression ϕ and every $a \in C^k$

$$R(a) = \inf_{b \in C^\ell} \phi^\Gamma(a, b).$$

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$\mathrm{Aut}(\Gamma)$ oligomorphic $\Rightarrow$ relations of Γ attain only finitely many values
$\Rightarrow$ infimum above is attained

Relations that do not change the complexity

Identify classical relations $R \subseteq C^k$ with functions $R\colon C^k \to \{0, \infty\}$.

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Fact: VCSP(Γ, R) reduces in poly-time to VCSP(Γ) whenever

- $k = 1$ and $R(a) = \emptyset$ for all $a \in C$,
- $k = 2$ and $R(a, b) = \{(a, b) \mid a = b\}$,
- R is expressible in Γ,
- $R = r \cdot S^\Gamma + s$ for some $S \in \tau$, $r \in \mathbb{Q}_{\geq 0}$ and $s \in \mathbb{Q}$,
- $R = \text{Feas}(S^\Gamma) := \{a \in C^k \mid S^\Gamma(a) < \infty\}$ for some $S \in \tau$,
- $R = \text{Opt}(S^\Gamma) := \{a \in \text{Feas}(S^\Gamma) \mid R(a) \leq R(b) \text{ for all } b \in C^k\}$ for some $S \in \tau$.

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Example: Let $R\colon \{0, 1, 2\} \to \{0, 1, \infty\}$ be defined by $R(0) = 0$, $R(1) = 1$ and $R(2) = \infty$. Then Feas(R) cannot be obtained from R by expressing, shifting, non-negative scaling and use of Opt and vice versa.

Relations that do not change the complexity

Identify classical relations $R \subseteq C^k$ with functions $R\colon C^k \to \{0, \infty\}$.

Fact: VCSP(Γ, R) reduces in poly-time to VCSP(Γ) whenever

- $k = 1$ and $R(a) = \emptyset$ for all $a \in C$,
- $k = 2$ and $R(a, b) = \{(a, b) \mid a = b\}$,
- R is expressible in Γ,
- $R = r \cdot S^\Gamma + s$ for some $S \in \tau$, $r \in \mathbb{Q}_{\geq 0}$ and $s \in \mathbb{Q}$,
- $R = \mathrm{Feas}(S^\Gamma) := \{a \in C^k \mid S^\Gamma(a) < \infty\}$ for some $S \in \tau$,
- $R = \mathrm{Opt}(S^\Gamma) := \{a \in \mathrm{Feas}(S^\Gamma) \mid R(a) \leq R(b) \text{ for all } b \in C^k\}$ for some $S \in \tau$.

Example: Let $R\colon \{0, 1, 2\} \to \{0, 1, \infty\}$ be defined by $R(0) = 0$, $R(1) = 1$ and $R(2) = \infty$. Then Feas(R) cannot be obtained from R by expressing, shifting, non-negative scaling and use of Opt and vice versa.

Denote the expansion of Γ that is closed under the operators above by $\langle \Gamma \rangle$.

Hardness from pp-constructions

Definition

- d-th pp-power of Γ: a valued structure Δ with domain C^d such that for every R of arity k in Δ there exists S of arity dk in $\langle\Gamma\rangle$ such that

$$R((a_1^1,\dots,a_d^1),\dots,(a_1^k,\dots,a_d^k)) = S(a_1^1,\dots,a_d^1,\dots,a_1^k,\dots,a_d^k).$$

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Example: $\Gamma_{q_\triangle}$ pp-constructs K_3 and therefore $\mathrm{RES}(q_\triangle)$ is NP-hard.

Fractional homomorphisms

Definition (fractional homomorphism)

- A fractional map from D to C is a probability distribution

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exists (always if Aut(Γ) is oligomorphic) and

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Remark: Fractional homomorphisms compose and hence we can define fractional homomorphic equivalence.

Outline

Fractional polymorphisms

polymorphism f of $\mathfrak{B}$ – $f : B^n \to B$ that preserves all relations of $\mathfrak{B}$

Example: The binary operation $+$ is a polymorphism of $(\mathbb{Q}, <)$.

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$$E_\omega[f \mapsto R(f(a^1, \dots, a^n))] \leq \frac{1}{n} \sum_{j=1}^{n} R(a^j) \quad (\omega \text{ improves } R).$$

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Remarks:

- Fractional polymorphisms of arity n of Γ are precisely fractional homomorphisms from a specific n-th pp-power Γ^n of Γ.
- Fractional polymorphisms of Γ with a countable domain improve all relations in $\langle \Gamma \rangle$.

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Id_n is a fractional polymorphism for every Γ since for every k-ary relation R and $a^1, \ldots, a^n \in C^k$

$$E_\omega[f \mapsto R(f(a^1, \ldots, a^n))] = \frac{1}{n} \sum_{i=1}^{n} R(\pi_i^n(a^1, \ldots, a^n)) = \frac{1}{n} \sum_{i=1}^{n} R(a^i).$$

Tractability in the finite-domain case

Definition (cyclic (fractional) operation)

- An operation $f : C^n \to C$, $n \geq 2$ is cyclic if for all $(x_1, \ldots, x_n) \in C^n$

$$f(x_1, \ldots, x_n) = f(x_2, \ldots, x_n, x_1).$$

- A fractional operation ω is called cyclic if for every $A \in B(C^{C^n})$

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Theorem

Γ – a finite-domain valued structure

- *If Γ does not pp-construct K_3, then Γ has cyclic fractional polymorphism (essentially Kozik, Ochremiak ('15)).*
- *If Γ has a cyclic fractional polymorphism, then $\text{VCSP}(\Gamma)$ is in P (Kolmogorov, Krokhin, Rolínek ('15)).*

Pseudo cyclic and canonical operations

Γ – valued structure with the domain C

Definition (pseudo cyclic, canonical operation)

An operation $f : C^n \to C$ for $n \geq 2$ is called

- pseudo cyclic with respect to $\mathrm{Aut}(\Gamma)$ if there are $e_1, e_2 \in \overline{\mathrm{Aut}(\Gamma)}$ such that for all $x_1, \ldots, x_n \in D$

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Canonicity and pseudo cyclicity for fractional operations is defined analogously as cyclicity.

Sufficient condition for tractability

Theorem (Bodirsky, Lutz, S.)

q – conjunctive query
If Γ_q has a fractional polymorphism which is canonical and pseudo cyclic with respect to $\mathrm{Aut}(\Gamma_q)$, then $\mathrm{VCSP}(\Gamma_q)$ is in P.

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- similar (but much more technical) for $\Gamma_{q_{\text{new}}}$

Tractability of $\mathrm{RES}(q_{\text{path}})$ and $\mathrm{RES}(q)$ was known in set semantics.

Outline

Tractability conjecture

Conjecture: If Γ_q does not pp-construct K_3, then Γ_q has a fractional polymorphism which is canonical and pseudo cyclic with respect to $\mathrm{Aut}(\Gamma_q)$ and hence $\mathrm{VCSP}(\Gamma_q)$ and $\mathrm{RES}(q)$ is in P.

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More ambitious – complexity dichotomy for VCSPs:
Conjecture: Let Γ be a valued structure with finite signature such that $\mathrm{Aut}(\Gamma) = \mathrm{Aut}(\mathfrak{B})$ for some reduct $\mathfrak{B}$ of a countable finitely bounded homogeneous structure. If K_3 has no pp-construction in Γ, then $\mathrm{VCSP}(\Gamma)$ is in P (otherwise, we already know that $\mathrm{VCSP}(\Gamma)$ is NP-complete).

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Generalizes the Bodirsky-Pinsker conjecture ('11) about infinite-domain CSPs and the dichotomy for finite-domain VCSPs.

Do fractional polymorphisms determine complexity?

Question: Let Γ be a valued structure with $\text{Aut}(\Gamma)$ oligomorphic. Is it true that $R \in \langle \Gamma \rangle$ if and only if R is improved by all fractional polymorphisms of Γ?

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- The implication from left to right is true.
- If the domain of Γ is finite, the answer is yes.
- If all relations are 0-∞ valued, the answer is yes.

Do we need integrals?

A fractional operation ω on C has finite support, if there are finitely many operations $f_1, \ldots, f_k$ on C such that $\sum_{i=1}^{k} \omega(f_k) = 1$.

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- And what if we restrict to structures of the form Γ_q for some query q?
- Is there a query q such that there exists a weighted relation R which is not improved by all fractional polymorphisms of Γ_q, but is improved by all fractional polymorphisms ω with finite support?

Thank you for your attention

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