Forbidden Tournaments and the Orientation (Completion) Problem

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Atlantic Graph Theory seminar

ERC Synergy Grant POCOCOP (GA 101071674)

General setting

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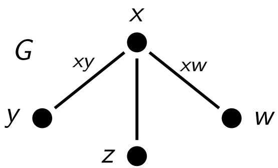
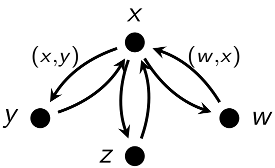
Simple graphs (possibly infinite)





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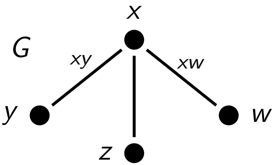
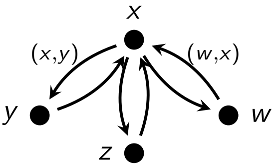
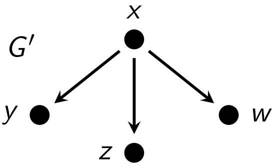
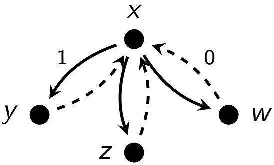




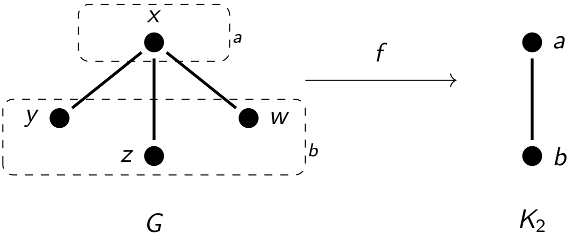
Oriented graphs (possibly infinite)





General setting

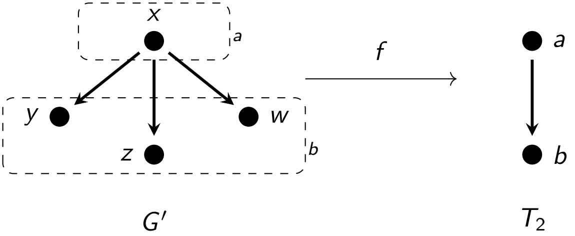
Homomorphism: Edge-preserving function $f\colon V(G) \to V(H)$



Observation: A graph G is k-colourable if and only if $G \to K_k$

General setting

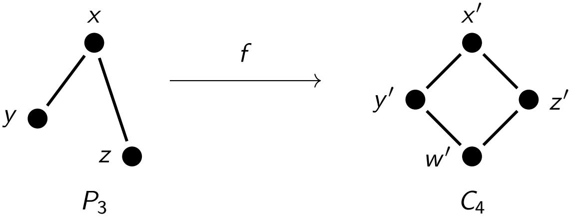
Homomorphism: Arc-preserving function $f: V(G) \to V(H)$



Observation: An oriented graph G' has a directed walk on k vertices if and only if $\overrightarrow{P}_k \to G'$

General setting

Embedding: Injective homomorphism $f\colon G \to H$ that preserves non-edges



Observation: $G < H$ if and only if H contains G as an *induced subgraph* (up to isomorphism)

General setting

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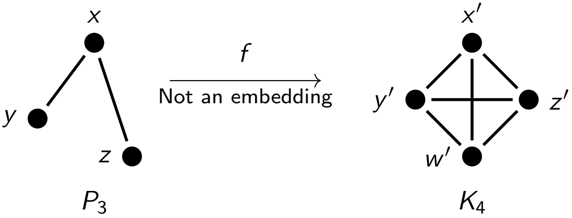

General setting

Hereditary class (property): A class $\mathcal{C}$ such that for each $G \in \mathcal{C}$ if $H < G$ then $H \in \mathcal{C}$

- Bipartite graphs
- Triangle-free graphs
- Forests
- H-colourable graphs
- Circular-arc graphs

F-free graphs: A graph G is $\mathcal{F}$-free if G does not contain any $F \in \mathcal{F}$ as induced subgraph

General setting

Theorem template (for hereditary classes): A graph G is a YYY-graph if and only if G is $\mathcal{F}_Y$-free

- Triangle-free graphs — K_3
- Forests — C_n with $n \geq 3$
- Perfect graphs — Odd holes and odd anti-holes
- Trivially-perfect graphs — C_4 and P_4
- Bipartite graphs — Odd cycle
- k-colourable graphs — Unknown for $k \geq 3$

Oriented expressions of graph classes

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Roy-Gallai-Hasse-Vitaver Theorem (60s)

A graph G is k-colourable if and only if there is an orientation G' of G with no directed walk on $k+1$ vertices.

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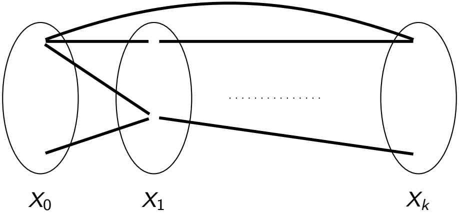

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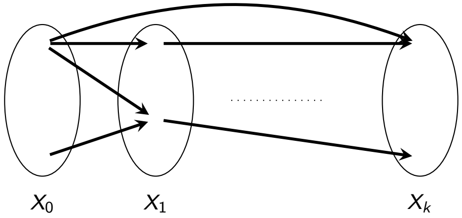

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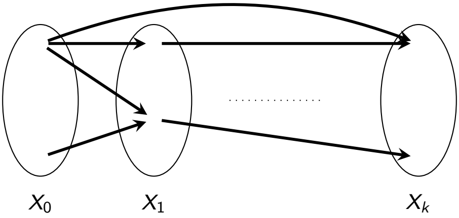

Conversely, let $c(v)$ be the length of the largest directed walk of G' starting in v. If $(x, y) \in E(G')$, then $c(x) > c(y)$. Thus, if $xy \in E(G)$, then $c(x) > c(y)$ or $c(y) > c(x)$.

Oriented expressions of graph classes

Comparability graphs

The comparability graph of a poset $(P, \leq)$ is the graph with vertex set P and $xy \in E$ if x and y are comparable in $(P, \leq)$.

A graph G is a comparability graph if G is the comparability graph of some poset P.

Theorem: A graph G is a comparability graph if and only if it is an F-free graph[1]

$$(XF_1^{2n+3}, XF_5^{2n+3}, XF_6^{2n+2}, \overline{C_{n+6}}, \overline{T_2}, \overline{X_2}, \overline{X_3}, \overline{X_{30}}, \overline{X_{31}}, \overline{X_{32}}, \overline{X_{33}}, \overline{X_{34}}, \overline{X_{35}}, \overline{X_{36}}, \text{co-}XF_2^{n+1}, \text{co-}XF_3^{n}, \text{co-}XF_4^{n}, \text{odd-hole})\text{-free}$$

[1] Screenshot: graphclasses.org

Oriented expressions of graph classes

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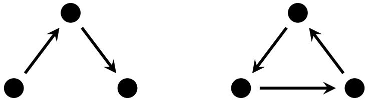
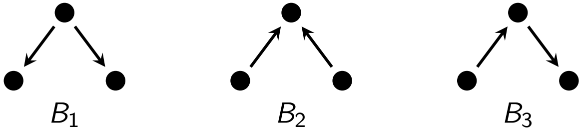
Observation: A graph G is a comparability graph if and only if it admits an $\mathcal{F}$-free orientation



Oriented expressions of graph classes

Skrien (1982)



- Comparability graphs
- Trivially perfect graphs
- Proper circular-arc graphs
- Perfectly orientable graphs

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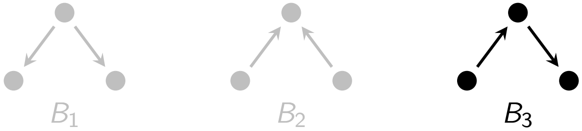

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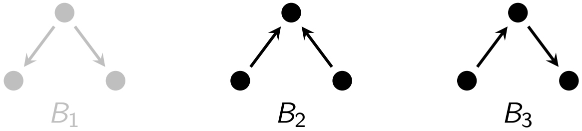

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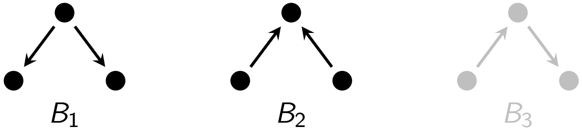

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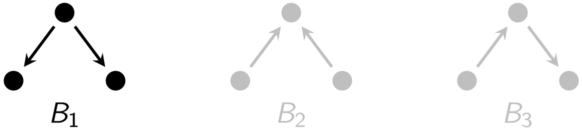

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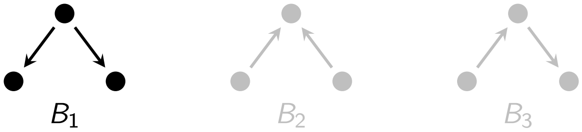


Perfectly orientable graphs

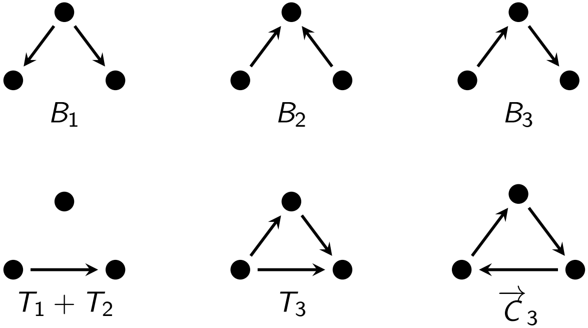
Gavril and Urrutia (1992): polynomial-time recognition algorithm
Hartinger and Milanic (2016–2017): towards structural characterization

General algorithm

Bang-Jensen and Gutin (2009): uniform reduction to 2-SAT

Oriented expressions of graph classes

Extending Skrien



Oriented expressions of graph classes

List of graph classes expressible by forbidden orientations on three vertices:

- Perfectly orientable graphs
- Comparability graphs
- Odd closed strip hom.-free graphs
- Proper circular-arc graphs
- Trivially perfect graphs
- Transitive-perfectly orientable graphs
- Unicyclic graphs
- Triangle-free unicyclic graphs
- 3-colourable comparability graphs
- Triangle-free graphs
- Clusters
- Proper Helly circular-arc graphs
- Triangle-free proper circular-arc graphs
- Paths and cycles
- Paths and cycles but no triangles
- Triangles and stars
- Star forests
- Stars and empty graphs
- Matchings with isolated vertices
- Empty graphs and K_2
- Bipartite graphs
- Complete bipartite graphs
- Complete 3-partite graphs
- $K_{2,3}$-free complete multipartite graphs
- Complete multipartite graphs
- All graphs

Oriented expressions of graph classes

Polynomial time recognition cases:

- Perfectly orientable graphs
- Comparability graphs
- Odd closed strip hom.-free graphs
- Proper circular-arc graphs
- Trivially perfect graphs
- Transitive-perfectly orientable graphs?
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Parallel research lines

Linear orderings: Damaschke (1990); Duffus, Ginn, and Rödl (1995); Hell, Mohar, and Rafiey (2014); Feuilloley and Habib (2021 –2023).

Circular orderings: Tucker (1972); G.P., Hell, and Hernández-Cruz (2023).

Tree-layouts: Paul and Protopapas (2023).

Vertex/edge colourings: Feder and Vardi (1999), Bodirsky, Madelaine, and Mottet (2021), Barsukov (2023), Bok, G.P., Hernández-Cruz, Jedličková (2024).

Three generic problems

Characterization Problem

Given a finite set of oriented graphs $\mathcal{F}$ characterize the class of graphs that admit an $\mathcal{F}$-free orientation (e.g., list their minimal obstructions).

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Given a finite set of oriented graphs $\mathcal{F}$ characterize the class of graphs that admit an $\mathcal{F}$-free orientation (e.g., list their minimal obstructions).

- Orientations of P_3 (Skrien 1982).
- Perfectly orientable graphs (Hartinger and Milanic 2016, 2017).
- Oriented graphs on 3 vertices (G.P. and Hernández-Cruz 2021).
- Open cases: $\mathcal{F} = \{B_1\}$ and $\mathcal{F} = \{B_1, \overrightarrow{C}_3\}$.

Three generic problems

Characterization Problem

Given a hereditary class of graphs $\mathcal{C}$, determine if there is a finite set of oriented graphs $\mathcal{F}$ such that, a graph G admits an $\mathcal{F}$-free orientation if and only if $G \in \mathcal{C}$.

Three generic problems

Characterization Problem

Given a hereditary class of graphs $\mathcal{C}$, determine if there is a finite set of oriented graphs $\mathcal{F}$ such that, a graph G admits an $\mathcal{F}$-free orientation if and only if $G \in \mathcal{C}$.

Positive results:

- Roy-Gallai-Hasse-Vitaver Theorem (1960 – 1968)
- C_{2n+1}-colourable graphs (G.P. and Hernández-Cruz 2021)
- $\overline{C_{2n+1}}$-colourable graphs (Gujgiczer and G.P. 2023+)
- Orientations *might be good* at distinguishing H-colourings

Three generic problems

Characterization Problem

Given a hereditary class of graphs $\mathcal{C}$, determine if there is a finite set of oriented graphs $\mathcal{F}$ such that, a graph G admits an $\mathcal{F}$-free orientation if and only if $G \in \mathcal{C}$.

Negative results:

- Forests
- Chordal graphs
- Even-hole free graphs
- Orientations *are bad* at distinguishing cycles (G.P. and Hernández-Cruz 22)

Three generic problems

Complexity Problem

Given a finite set of oriented graphs $\mathcal{F}$, determine the complexity of deciding if an input graph G admits an $\mathcal{F}$-free orientation

- In P when $\mathcal{F}$ is a set of oriented graphs on 3 vertices (Urrutia and Gavril 1992, Bang-Jensen and Gutin 2007, G.P. and Hernández-Cruz 2021).
- Open case: $\mathcal{F} = \{B_1, \overrightarrow{C}_3\}$.

B_1 $\overrightarrow{C}_3$

Three generic problems

Complexity Problem (completion version)

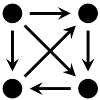
Given a finite set of oriented graphs $\mathcal{F}$, determine the complexity of deciding if an input partially oriented graph G can be completed to an $\mathcal{F}$-free oriented graph?

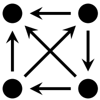
Three generic problems

Complexity Problem (completion version)

Given a finite set of oriented graphs $\mathcal{F}$, determine the complexity of deciding if an input partially oriented graph G can be completed to an $\mathcal{F}$-free oriented graph?

- Orientations of P_3 always in P (Bang-Jensen, Huang, Zhu, 2017).
- T_3-free orientation completion problem in P.
- (Bang-Jensen, Huang, Zhu) NP-complete for:





Three generic problems

Task for today

Given a finite set of tournaments $\mathcal{F}$:

1. Understand the complexity of the $\mathcal{F}$-free orientation problem.
2. Understand the complexity of the $\mathcal{F}$-free orientation completion problem.

Computational complexity

Computational complexity

Boolean satisfiability problem

Input: $(x_1^1 \vee x_2^1 \vee \cdots \vee x_{i_1}^1) \wedge \cdots \wedge (x_1^n \vee x_2^n \vee \cdots \vee x_{i_n}^n)$
Question: Is the instance satisfiable?

First known problem to be NP-complete (Cook 1969, Levin 1970)

k-SAT
Input: $(x_1^1 \vee x_2^1 \vee \cdots \vee x_k^1) \wedge \cdots \wedge (x_1^n \vee x_2^n \vee \cdots \vee x_k^n)$
Question: Is the instance satisfiable?

In P if $k \leq 2$, and otherwise NP-complete.

Computational complexity

Ladner's Theorem (1975)

If P$\neq$ NP, then there are NP-intermediate problems, i.e., problems in NP which are not solvable in polynomial-time nor NP-complete.

Candidate for NPI: Graph isomorphism problem.

Computational complexity

NAE 3-SAT
Input: $(x_1 \vee y_1 \vee z_1) \wedge \cdots \wedge (x_n \vee y_n \vee z_n)$
Question: Is there a solution such that $(x_i, y_i, z_i) \notin \{(0,0,0),(1,1,1)\}$?

Horn-SAT
Input: $(\neg x_1^1 \vee x_2^1 \vee \cdots \vee x_k^1) \wedge \cdots \wedge (\neg x_1^n \vee x_2^n \vee \cdots \vee x_k^n)$
Question: Is the instance satisfiable?

1-in-3 SAT
Input: $(x_1 \vee y_1 \vee z_1) \wedge \cdots \wedge (x_n \vee y_n \vee z_n)$
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Computational complexity

Linear equations (mod 2)
Input: $(x_1 \vee y_1 \vee z_1) \wedge \cdots \wedge (x_n \vee y_n \vee z_n)$
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Computational complexity

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Scheafer's dichotomy theorem (moral version)
Every Boolean satisfaction problem CSP($\mathbb{B}$) is either in P or NP-complete. Moreover, if it is not NP-complete, then CSP($\mathbb{B}$) is equivalent to one of the following cases:

- trivial-SAT
- Horn-SAT
- 2-SAT
- linear equations modulo 2

Classifications in graph theory

Folk: The k-colouring problem is in P if $k \leq 2$, and otherwise it is NP-complete

Hell-Nešetřil (1990): If H is a finite graph, then the H-colouring problem is in P if H is bipartite, otherwise it is NP-complete.

Barto, Kozik, Niven (2009): If D is a (core) digraph with no sources nor sinks, then CSP(D) if in P if every component of D is a directed cycle, and NP-complete otherwise (conjectured by Bang-Jensen and Hell 1990).

CSP dichotomy (Bulatov 2017, Zhuk 2017): If D is a finite digraph, then CSP(D) is either polynomial-time solvable or NP-complete (conjectured by Feder and Vardi 1999).

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Classifications in graph theory

Example 1: For every k there is a countable graph $\mathbb{H}_k$ such a finite graph is an induced subgraph of $\mathbb{H}_k$ if and only if G is K_k-free. In particular, $G \to \mathbb{H}_k$ if and only if G is K_k-free.

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Example 2: Consider the digraph $(\mathbb{Q}, <)$, i.e., $V = \mathbb{Q}$ and $(p, q) \in E$ if and only if $p < q$. Then, $D \to (\mathbb{Q}, <)$ if and only if D has no directed cycle.

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Tractability conjecture (Bodirsky-Pinsker 2011): If D is a countable digraph and [redacted], then CSP(D) is either polynomial-time solvable or NP-complete.

Classifications in graph theory

Fact (Fraïssé's theorem): For every finite set of tournaments $\mathcal{F}$ there is an infinite graph $D_{\mathcal{F}}$ such that an oriented graph G' is $\mathcal{F}$-free if and only if it is an induced oriented subgraph of $D_{\mathcal{F}}$.

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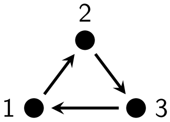
Task of today: Is there dichotomy for CSP($H_{\mathcal{F}}$)?

The $\mathcal{F}$-free orientation problem

Example 1: Every tournament in $\mathcal{F}$ has a directed cycle



$$\mathcal{F} = \{\overrightarrow{C}_3\}$$

The $\mathcal{F}$-free orientation problem

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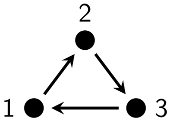


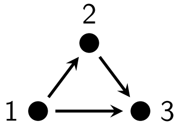
$$\mathcal{F} = \{\overrightarrow{C}_3\}$$

Remark: $\mathcal{F}$-free orientation problem is trivial

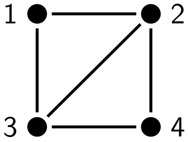
But: Orientation completion not necessarily trivial.

The $\mathcal{F}$-free orientation problem

Example 2: T_3-free orientation (completion) problem.



$\mathcal{F} = \{T_3\}$



G

Code orientations of G as solutions to the sys. lin. eq. over $\mathbb{Z}_2$

$$x_{ij} + x_{ji} = 0 \text{ for } ij \in E$$

The $\mathcal{F}$-free orientation problem

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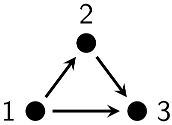

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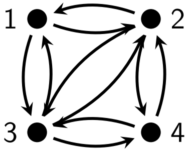

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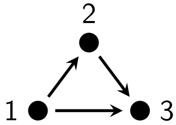

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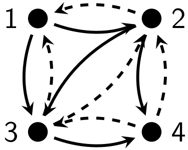
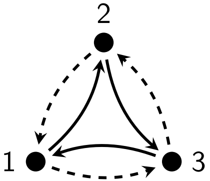


G'

$$x_{12} = 1,\ x_{13} = 1,\ x_{23} = 1,\ x_{24} = 1,\ x_{34} = 1$$
$$x_{21} = 0,\ x_{31} = 0,\ x_{32} = 0,\ x_{42} = 0,\ x_{43} = 0$$

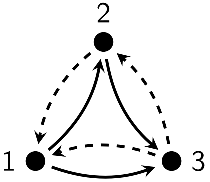
The $\mathcal{F}$-free orientation problem



$\overrightarrow{C}_3$

For each triangle i, j, k the following equality holds:

$$x_{ij} + x_{jk} = 0.$$

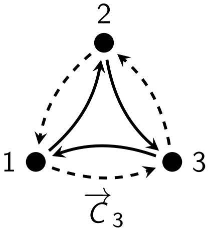


T_3

There exists a triangle i, j, k such that the following equality holds:

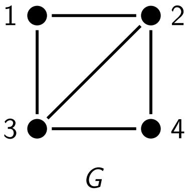
$$x_{ij} + x_{jk} = 1 \text{ for instance } x_{23} + x_{31} = 1.$$

The $\mathcal{F}$-free orientation problem



For each triangle i, j, k the following equality holds:

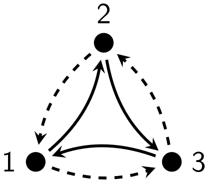
$$x_{ij} + x_{jk} = 0.$$



Code T_3-free orientation of G as solutions to

$$x_{ij} + x_{ji} = 0 \text{ for } ij \in E$$
$$x_{ij} + x_{jk} = 0 \text{ for } ijk \in T$$

The $\mathcal{F}$-free orientation problem



$\overrightarrow{C}_3$

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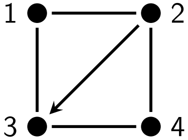
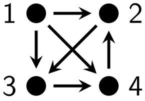


G

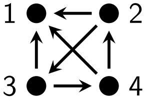
Code T_3-free orientation completions of G as solutions to

$$x_{ij} + x_{ji} = 0 \text{ for } ij \in E$$
$$x_{ij} + x_{jk} = 0 \text{ for } ijk \in T$$
$$x_{ij} = 1 \text{ for } ij \in A$$

The $\mathcal{F}$-free orientation problem



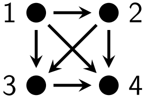
C_3^-



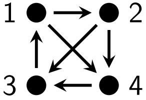
C_3^+

For each i, j, k, l in C_3^- and in C_3^+

$$x_{ij} + x_{jk} + x_{kl} + x_{li} = 1.$$



T_4



TC_4

$$x_{12} + x_{24} + x_{43} + x_{31} = 0 \text{ in } T_4$$
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The $\mathcal{F}$-free orientation problem

Example 3: The $\{T_4, TC_4\}$-free orientation (completion) problem is in P

The $\mathcal{F}$-free orientation problem

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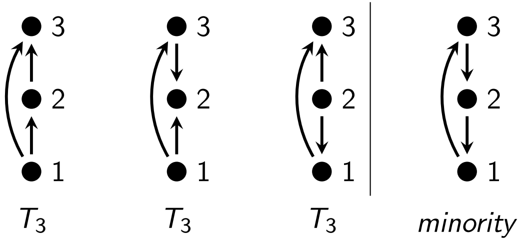
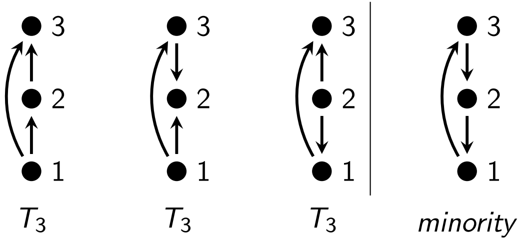
Question: For which finite sets of tournaments $\mathcal{F}$ the $\mathcal{F}$-free does this method work?

The $\mathcal{F}$-free orientation problem



- $\overrightarrow{C}_3$-free tournaments **are not** *preserved* by the *minority* operation.
- T_3-free tournaments **are** *preserved* by the *minority* operation.

The $\mathcal{F}$-free orientation problem



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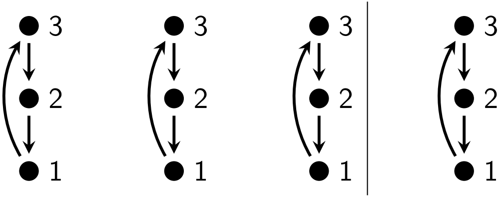

$\overrightarrow{C}_3$ $\overrightarrow{C}_3$ $\overrightarrow{C}_3$ *minority*

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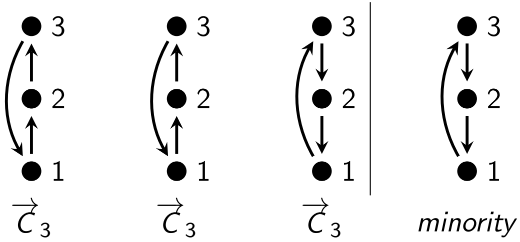

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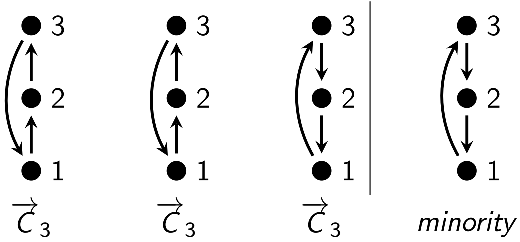

Lemma

Let $\mathcal{F}$ be a finite set of tournaments. The $\mathcal{F}$-free orientations of a graph G correspond to the solution space of some system of linear equations if and only if the $\mathcal{F}$-free tournaments are preserved by the minority operation.

The $\mathcal{F}$-free orientation problem

Example 4: The $\overrightarrow{C}_3$-free orientation completion problem is NP-complete

The $\mathcal{F}$-free orientation problem

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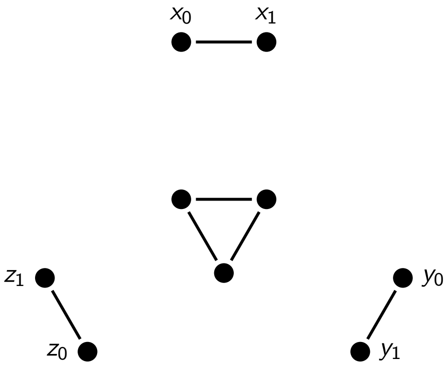
Reduction from NAE 3-SAT with Input: $(x \vee y \vee z) \wedge \ldots$



The $\mathcal{F}$-free orientation problem

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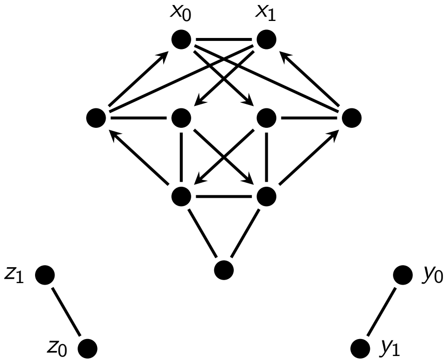

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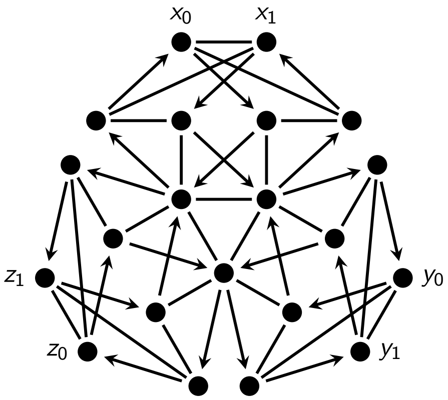


The $\mathcal{F}$-free orientation problem

Is that all?

The $\mathcal{F}$-free orientation problem

Theorem (Bodirsky, G.P., 23+)

For every finite set of finite tournaments $\mathcal{F}$ one of the following cases holds.

1. $\mathcal{F}_f$ is preserved by the minority operation. In this case, the $\mathcal{F}$-free orientation completions of a partially oriented graph G correspond to the solution space of a system of linear equations over $\mathbb{Z}_2$.
2. Otherwise, $\mathcal{F}$-free orientation completion problem is NP-complete.

In the first case, the $\mathcal{F}$-free orientation completion problem is in P.

The $\mathcal{F}$-free orientation problem

Corollary

If every tournament in $\mathcal{F}$ contains a directed cycle, then the $\mathcal{F}$-free orientation completion problem is NP-complete.

(Particular instance previously considered by Bang-Jensen, Huang, and Zhu).

The $\mathcal{F}$-free orientation problem

Theorem (Bodirsky, G.P., 23+)

For every finite set of finite tournaments $\mathcal{F}$ one of the following cases holds.

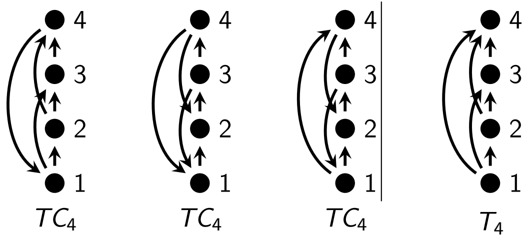
1. $\mathcal{F}$ contains no transitive tournament. In this case, every graph admits an $\mathcal{F}$-free orientation.
2. $\mathcal{F}_f$ is preserved by the minority operation. In this case, the $\mathcal{F}$-free orientations of a graph G correspond to the solution space of a system of linear equations over $\mathbb{Z}_2$.
3. Otherwise, $\mathcal{F}$-free orientation problem is NP-complete.

In cases 1 and 2, the $\mathcal{F}$-free orientation problem is in P.

The $\mathcal{F}$-free orientation problem

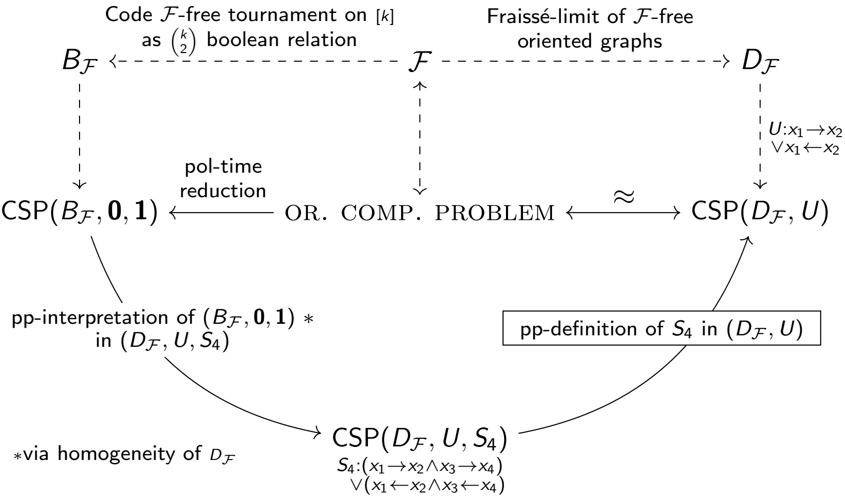
Corollary

The T_k-free orientation problem is NP-complete for each $k \geq 4$.



If the $\mathcal{F}$-free orientaion problem is NP-hard, then it is still NP-hard for K_f-free graphs.

Proof overview



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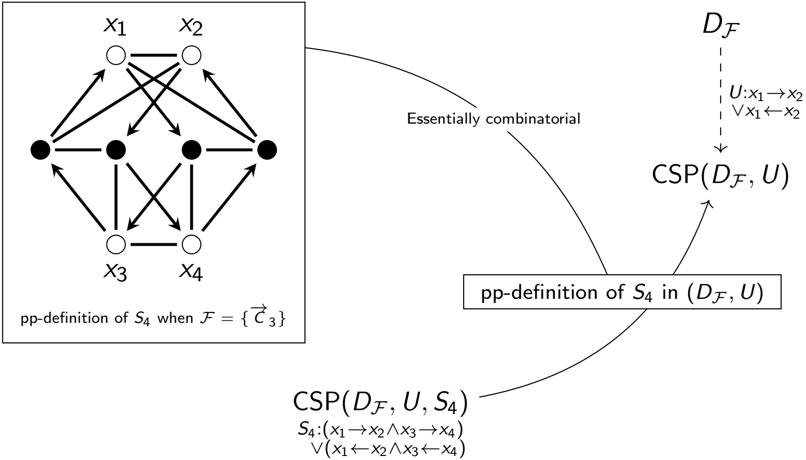

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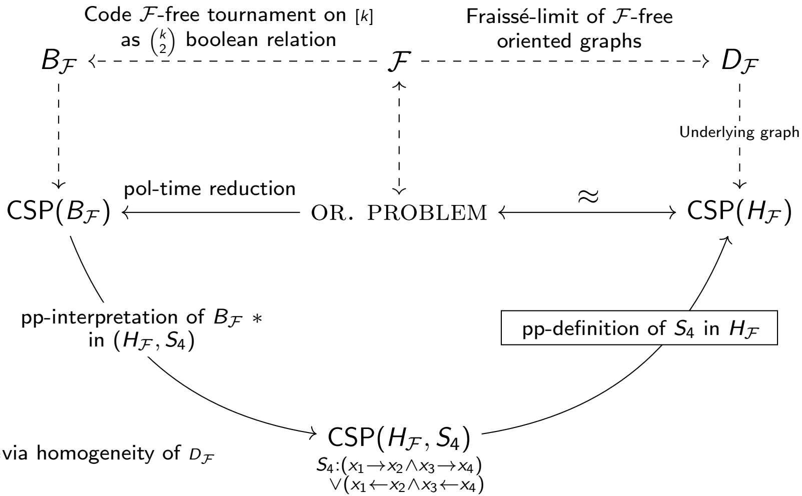
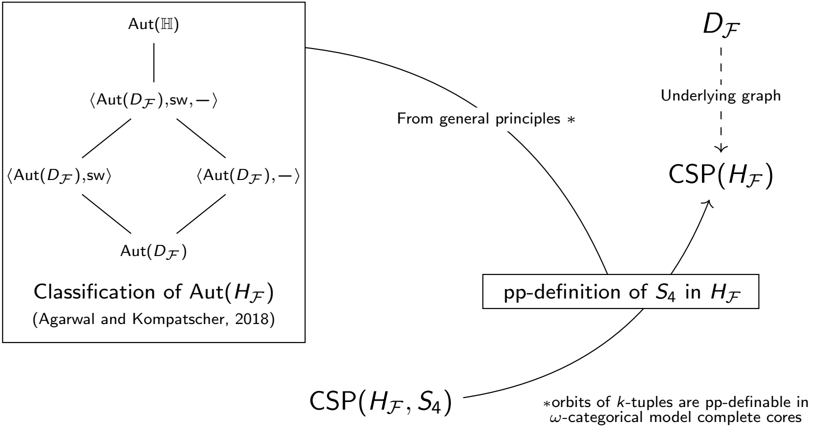


*via homogeneity of $D_{\mathcal{F}}$

Proof overview



Thank you!

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