TECHNISCHE UNIVERSITÄT WIEN

Measures in homogeneous 3-hypergraphs

Or: Why we cannot have good things because we have everything

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Outline

Invariant Keisler measures

We work in a first-order structure $\mathcal{M}$ which is ω-saturated and strongly ω-homogeneous (true for a monster model or $\mathcal{M}$ countable ω-categorical).

Definition 1 (Keisler measure)

A **Keisler measure** on $\mathcal{M}$ in the variable x is a finitely additive **probability** measure on $\mathrm{Def}_x(M)$:

- $\mu(X \cup Y) = \mu(X) + \mu(Y)$ for disjoint X and Y;
- $\mu(M) = 1$.

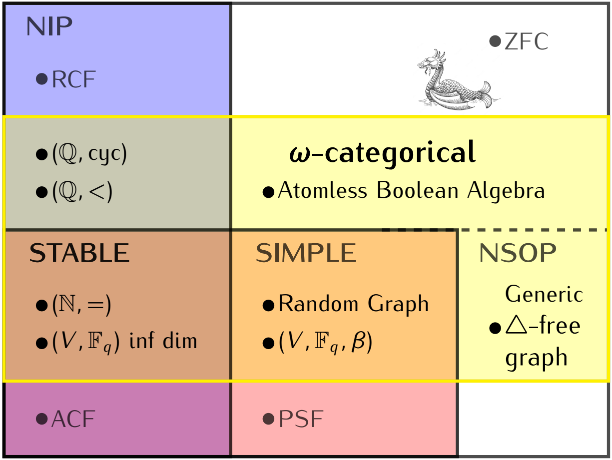
We want to study Keisler measures **invariant** under automorphisms. We call these **invariant Keisler measures** (IKMs):

$$\mu(X) = \mu(\sigma \cdot X) \text{ for } \sigma \in \mathrm{Aut}(M),$$

where $\sigma \cdot \phi(M, \overline{a}) = \phi(M, \sigma(\overline{a}))$.

ω-categorical structures

$\mathcal{M}$ is ω-categorical when its theory has a unique countable model up to isomorphism.



Homogeneous structures

All of our examples are **homogeneous**: any isomorphism between finite substructures extends to an automorphism of the whole structure.

When a class of finite structures $\mathcal{C}$ forms a **Fraïssé class** we can build a countable homogeneous structure $\mathcal{M}$ whose **age**, i.e. its class of finite substructures, is $\mathcal{C}$. We call $\mathcal{M}$ the **Fraïssé limit** of $\mathcal{C}$.

Examples 2

Homogeneous structure	Fraïssé class
Random graph	finite graphs
Generic $\triangle$-free graph	finite $\triangle$-free graphs
Universal homogeneous 3-hypergraph	finite 3-hypergraphs

MS-measurable structures

An MS-measurable structure (Macpherson & Steinhorn 2007) has a dimension-measure function h assigning each definable set a dimension and a measure satisfying various desirable properties

More on MS-measurable structures

What you need to know for this talk:

- MS-measurable $\Rightarrow$ supersimple of finite SU-rank;
- In an MS-measurable structure, any A-definable set induces an $\mathrm{Aut}(M/A)$-invariant measure on its definable subsets;
- This measure assigns a **positive** value to every definable subset non-forking over A;
- **Examples:** pseudofinite fields, the random graph, infinite dimensional vector spaces over finite fields, etc.

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Stationarity

Theorem 3 (Stationarity, Hrushovski 2012)

Say $\mathrm{acl}^{eq}(\emptyset) = \mathrm{dcl}^{eq}(\emptyset)$. *Then, for* μ *an IKM and* $a \mathop{\smile\!\!\!\!\!\!\mid}\, b$,

$$\mu(\phi(x,a) \wedge \psi(x,b))$$

only depends on $\mathrm{tp}(a)$ *and* $\mathrm{tp}(b)$, *but not on* $\mathrm{tp}(ab)$.

- Proof follows from the fact $\mu(\phi(x,a) \wedge \psi(x,b)) = \alpha$ is **stable** (Hrushovski 2012)+standard stability theory;
- If $\mathcal{M}$ is ω-categorical, $a \mathop{\smile\!\!\!\!\!\!\mid}\, b$ can be replaced with $\mathrm{acl}^{eq}(a) \cap \mathrm{acl}^{eq}(b) = \mathrm{acl}^{eq}(\emptyset)$ (Jahel & Tsankov 2022).

Uses of stationarity

- It allows for Szemeredi regularity results in pseudofinite fields and other MS-measurable structures (Pillay & Starchenko 2013).

If $\mathcal{M}$ is the countable model of an ω-categorical theory,

- for an **ergodic** measure μ and $\mathrm{acl}^{eq}(a) \cap \mathrm{acl}^{eq}(b) = \mathrm{acl}^{eq}(\emptyset)$,

$$\mu(\phi(x,a) \wedge \psi(x,b)) = \mu(\phi(x,a))\mu(\psi(x,b));$$

- every invariant measure μ can be written an an integral average of the ergodic measures:

$$\mu(X) = \int_{\mathrm{Erg}_x(M)} \nu(X) \mathrm{d}\nu.$$

So stationarity is extremely helpful in understanding the invariant Keisler measures of a structure.

Example: measures in homogeneous graphs

Theorem 4 (Measures on the Random graph, Albert 1990)

Let μ be an IKM for the random graph R (in the variable x). Then, there is a unique measure ν on $[0,1]$ such that

$$\mu(\phi(x, A, B)) = \int_0^1 p^{|A|}(1-p)^{|B|}\mathrm{d}\nu,$$

where for finite and disjoint $A, B \subseteq R, \phi(x, A, B)$ asserts that x is connected to all of A and none of B.

Theorem 5 (Measures on the generic $\triangle$-free graph, Albert 1990)

The generic triangle free graph has a unique IKM corresponding to the unique invariant type p asserting that x is disconnected from everything.

Understanding measures in higher arity

Stationarity gives a powerful tool to understand measures in binary structures.

Question 1

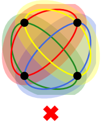
Can we also understand measures of more complex intersections? E.g. how can we study the measure of a formula of the form

$$\phi(w, ab) \wedge \psi(w, ac) \wedge \chi(w, bc)?$$

Attack strategy: look at IKMs in homogeneous 3-hypergraphs.

Three case studies

- **Universal homogeneous $\mathbf{3}$-hypergraph** $\mathcal{R}_3$;
  A **$\mathbf{3}$-hypergraph** has a ternary hyperedge relation taking distinct triplets of vertices.
- **Generic tetrahedron-free $\mathbf{3}$-hypergraph** $\mathcal{H}$;
  A **tetrahedron** consists of four vertices such that each three of them form a hyperedge.



- **Universal homogeneous two-graph** $\mathcal{G}$;
  A **two-graph** is a 3-hypergraph such that any four vertices have an even number of hyperedges.

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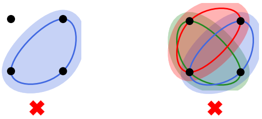

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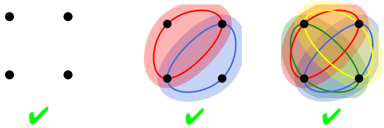


Standard properties

Theorem 6 (Basic properties, folklore (+Koponen 2017))

Let $\mathcal{M} = \mathcal{R}_3, \mathcal{H}$, or $\mathcal{G}$:

- *For any $A \subset M$, $\mathrm{acl}(A) = A$ and $\mathrm{acl}^{eq}(A) = \mathrm{dcl}^{eq}(A)$;*
- *$\mathcal{M}$ is **simple** with **trivial independence**:*

$$A \underset{C}{\downarrow} B \text{ if and only if } A \cap C = A \cap (B \cup C);$$

- *$\mathcal{M}$ is **one-based**, i.e. for any $A, B \subseteq M^{eq}$,*

$$A \underset{\mathrm{acl}^{eq}(A) \cap \mathrm{acl}^{eq}(B)}{\downarrow} B.$$

Moreover, $\mathcal{R}_3$ and $\mathcal{G}$ are known to be MS-measurable (Macpherson & Steinhorn 2007).

Higher stationarity?

Question 2 (Higher Stationarity)

Let $\mathcal{M}$ be ω-categorical. Given adequate independence conditions on a, b, c, do we have that for formulas $\phi(w, ab), \psi(w, ac), \chi(w, bc)$ and an invariant Keisler measure μ,

$$\mu(\phi(w, ab) \wedge \psi(w, ac) \wedge \chi(w, bc))$$

only depends on the types of the pairs ab, ac, bc and not on the type of the triplet?

- would be helpful for classifying measures;
- holds for various ω-categorical structures (e.g. homogeneous graphs);
- inspired also by hypergraph regularity results in pseudofinite fields (Chevalier & Levi 2022).

NIP+ω-categorical $\Rightarrow$ strong higher stationarity

Theorem 7

Let $\mathcal{M}$ be NIP and ω-categorical with $\mathrm{acl}^{eq}(\emptyset) = \mathrm{dcl}^{eq}(\emptyset)$. Given triplets of tuples abc and $a'b'c'$ agreeing on the types of pairs:

$$\mu(\phi(w, ab) \wedge \psi(w, ac) \wedge \chi(w, bc)) = \mu(\phi(w, a'b') \wedge \psi(w, a'c') \wedge \chi(w, b'c')).$$

Proof.

μ must be an **integral average of invariant types** (Braunfeld & M. 2023, Hrushovski & Pillay 2011).
Then, just note that one formula is in an invariant type if and only if the other is.

□

Higher stationarity in an MS-measurable context

Let $\mathcal{M}$ be ω-categorical, abc be an independent triplet and $e \downarrow abc$. Write $\phi(x, ab), \psi(x, ac), \chi(x, bc)$ for the formulas isolating $\mathrm{tp}(e/ab), \mathrm{tp}(e/ac), \mathrm{tp}(e/bc)$.

Theorem 8 (Independence in an MS-measure, M. 2023)

Let $\mathcal{M}$ be ω-categorical and MS-measurable with $\mathrm{acl}^{eq}(\emptyset) = \mathrm{dcl}^{eq}(\emptyset)$ and $\mathrm{acl}^{eq}(e) = \mathrm{dcl}^{eq}(e)$. Suppose that for any independent triplet $a'b'c'$ agreeing with abc on the types of pairs,

$$h(\phi(w, ab) \wedge \psi(w, ac) \wedge \chi(w, bc)) = h(\phi(w, a'b') \wedge \psi(w, a'c') \wedge \chi(w, b'c')),$$

where h is the MS-dimension-measure. Then,

$$\mu(\phi(w, ab) \wedge \psi(w, ac) \wedge \chi(w, bc)) = \frac{\mu(e/ab)\mu(e/ac)\mu(e/bc)\mu(e)}{\mu(e/a)\mu(e/b)\mu(e/c)}.$$

Higher stationarity in simple theories?

Question 3

Does higher stationarity hold in ω-categorial **simple theories**?

Answer: No.
Counterexample: the universal homogeneous two-graph.

A unique measure on $\mathcal{G}$

Theorem 9 (A unique measure for $\mathcal{G}$, M. 2023)

$\mathcal{G}$ has a unique invariant Keisler measure μ in the singleton variable. For $d, a_1, \ldots, a_n \in G$ distinct, we have that

$$\mu(d/a_1 \ldots a_n) = \left(\frac{1}{2}\right)^{n-1},$$

and $\mu(d) = 1$.

Proof idea.

Fixing $c \in \mathcal{G}$, $(\mathcal{G}, c)$ is essentially a copy of the random graph. Hence, an IKM μ on $\mathcal{G}$ induces an IKM on the random graph μ'.
But IKMs on the random graph are classified!
Other equations satisfied by μ force a unique choice for μ' and so a unique choice for μ. □

Some remarks on the theorem

- **Something unusual:** $\mathcal{G}$ has a unique invariant measure, but no invariant types (in spite of $\mathrm{acl}^{eq}(\emptyset) = \mathrm{dcl}^{eq}(\emptyset)$).
- Another example of this: any infinite dimensional vector space over a finite field with a symplectic bilinear form.
- The result can also be deduced from (Basso & Jahel 2021), who proves that $\mathcal{G}$ is **uniquely ergodic**. Usually, unique ergodicity says nothing on the space of invariant Keisler measures.

Failure of higher stationarity

Take $a, b, c \in \mathcal{G}$ such that $\neg T(a, b, c)$. Then,

$$\mu(T(x, ab) \wedge T(x, ac) \wedge T(x, bc)) = 0,$$

because if there was any such x, there would be four vertices with three hyperedges.

Meanwhile, for $a', b', c' \in \mathcal{G}$ such that $T(a', b', c')$,

$$\mu(T(x, a'b') \wedge T(x, a'c') \wedge T(x, b'c')) = \frac{1}{4}.$$

Hence, **higher stationarity can fail** in simple theories!

Non-MS-measurability of the generic tetrahedron-free 3-hypergraph

Theorem 10 (M. 2023)

The generic tetrahedron-free 3-hypergraph $\mathcal{H}$ is not MS-measurable.

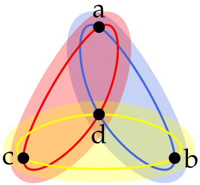
Previously, there were no known examples of supersimple one-based ω-categorical structures which were not MS-measurable.

Indeed, the only previously known examples come from ω-categorical Hrushovski constructions (Evans 2022, M. 2022) and made heavy use of the assumption of not being one-based.

Forcing higher stationarity

The idea behind the proof is **forcing higher stationarity**, which allows for easier calculations with an MS-measure (given the earlier theorem).

Fix a vertex d and consider a, b, c such that each pair forms a hyperedge with d. This implies $\neg T(a, b, c)$ since otherwise, $abcd$ would be a tetrahedron!



Hence, there is a unique independent 3-type over d for which the types of pairs over d agree with abc.

Getting equations from the forced higher stationarity

Given $e \mathop{\smile\!\!\!\!\!\!\,|}\, abcd$, take $\phi(x, ab, d), \psi(x, ac, d), \chi(x, bc, d)$ isolating $\mathrm{tp}(e/abd), \mathrm{tp}(e/acd), \mathrm{tp}(e/bcd)$ respectively.
From our earlier theorem, the measure μ of an MS-dimension measure satisfies

$$\mu(\phi(x, ab, d) \wedge \psi(x, ac, d) \wedge \chi(x, bc, d)) = \frac{\mu(e/abd)\mu(e/acd)\mu(e/bcd)\mu(e/d)}{\mu(e/ad)\mu(e/bd)\mu(e/cd)}.$$

Various equations jointly imply that $\mu(T(x, ab)) = 0$, which contradicts **positivity** of an MS-dimension-measure.

A conjecture on the universal homogeneous 3-hypergraph

What about IKMs on $\mathcal{R}_3$?

Conjecture 11 (Ensley 2001)

Every invariant Keisler measure on $\mathcal{R}_3$ is of the form

$$\mu(\Phi(x, A, B)) = \int_0^1 p^{|B|}(1-p)^{\binom{n}{2}-|B|}\mathrm{d}\nu,$$

where for $|A| = n, B \subseteq [A]^2$, $\Phi(x, A, B)$ expresses that x forms a hyperedge with each pair in B and does not form a hyperedge with each pair in $[A]^2 \setminus B$.

Structure-independent measures

Definition 12 (Structure-independent measure)

Given $\Phi(x, A, B)$, we can draw a graph $G_{A,B}$ on $\{1, \ldots, n\}$ for $A = \{a_1, \ldots, a_n\}$, where

$$G \vDash E(i, j) \text{ if and only if } \{a_i, a_j\} \in B.$$

We say that an invariant Keisler measure μ on $\mathcal{R}_3$ is **structure-independent** if $\mu(\Phi(x, A, B))$ is entirely determined by the isomorphism type of $G_{A,B}$ as a graph.

Remark 13

Structure-independent measures satisfy higher stationarity. Can we at least understand these?

Cameron measures

Let $\mathcal{M}$ be an infinite relational structure. Consider $\mathrm{Age}(M)$ as a tree:

- **nodes at level n :** structures in $\mathrm{Age}(M)$ with vertex set $[n]$;
- **parents of a node at level n :** induced substructures on $[n-1]$

Definition 14

A **Cameron measure** on M corresponds to an isomorphism-invariant function $p : \mathrm{Age}(M) \to [0,1]$ such that

1. $p(\emptyset) = 1$;
2. $p(A) = \sum\{p(A') | A'$ is a child of $A\}$;

"The defect of the general approach is, however, that there are too many solutions, and they have no obvious connection with the structure under consideration" –Cameron 1990

S_∞-invariant measures on Struc_L

Definition 15

Let L be a countable relational language. Struc_L is the space of L-structures with domain ω. It has a topology with a basis of clopen sets given by

$$[\![\phi(\overline{a})]\!] = \{M \in \mathrm{Struc}_L | M \vDash \phi(\overline{a})\},$$

where $\phi(\overline{x})$ is a quantifier-free L-formula and $\overline{a}$ is a tuple from ω of length $|\overline{x}|$.

Definition 16

An **S_∞-invariant measure on Struc_L** is a Borel probability measure on Struc_L invariant under the action of S_∞.

These measures have been heavily studied, especially in a series of papers by Ackermann, Freer, Patel and collaborators.

When do S_∞-invariant measures concentrate on a structure?

Theorem 17 (AFP 2016)

Let $\mathcal{M}$ be a countable L-structure. Tfae:

- *there is an S_∞-invariant measures on Struc_L concentrated on the isomorphism type of $\mathcal{M}$.*
- *$\mathcal{M}$ has trivial group-theoretic definable closure, i.e. for any finite $A \subseteq M$,*

$$A = \mathrm{DCL}(A) = \{b \in M | \forall \sigma \in \mathrm{Aut}(M/A), \sigma(b) = b\}.$$

Some correspondences

Lemma 18 (Braunfeld & M. 2023)

There is a one-to-one correspondence between structure independent IKMs on $\mathcal{R}_3$ and Cameron measures on the random graph R.

Given an S_∞-invariant measure ν on Struc_L, the **Age of ν**, $\mathrm{Age}(\nu)$ is the class of finite $\mathcal{L}$-structures whose quantifier-free type $\phi(\overline{x})$ is assigned positive measure (i.e. such that $\nu(\phi(1, \dots, n)) > 0$ for $|\overline{x}| = n$).

Lemma 19 (M. 2023)

There is a one-to-one correspondence between Cameron measures on R and S_∞-invariant measures ν on Struc_E with $\mathrm{Age}(\nu) \subseteq \mathrm{Age}(R)$.

Structure-independent measures on $\mathcal{R}_3$

Corollary 20 (M. 2023)

There is a one-to-one correspondence between structure independent IKMs on $\mathcal{R}_3$ and S_∞-invariant measures on Struc_E concentrating on graphs.
Moreover, there is also a one-to-one correspondence between the ergodic measures in the two spaces of measures.

In particular, for ν an S_∞-invariant measure on Struc_E concentrating on graphs, we obtain a unique IKM μ on $\mathcal{R}_3$ by setting

$$\mu(\Phi(x, A, B)) := \nu([\![\Psi_{A,B}(1, \ldots, n)]\!]),$$

where $\Psi_{A,B}(x_1, \ldots, x_n)$ isolates the quantifier-free type of $G_{A,B}$ on $\{1, \ldots, n\}$.

Why we cannot have good things because we have everything

We know that the space of S_∞-invariant measures on Struc_E concentrated on graphs is extremely large. For example,

- any countable graph with trivial group-theoretic definable closure induces an S_∞-invariant measure on Struc_E (AFP 2016). This includes
  - the random graph;
  - any universal homogeneous K_n-free-graph for $n \geq 3$;
  - any countable universal C-free graph where C is a finite homomorphism-closed class of finite connected graphs.

  Except for trivial cases, for each such graph $\mathcal{M}$ there are continuum-many ergodic measures concentrated on $\mathcal{M}$ (AFP & Kwiatkowska 2017).
- there are also ergodic measures on Struc_E not concentrating on any S_∞-orbit, e.g. random geometric graphs (AFP & Kruckman 2017, Balister et al. 2018).

More general correspondences (for a future talk)

These correspondences can be pushed quite far for homogeneous ω-categorical structures (with $\mathrm{Aut}(M)$ acting transitively on M):

$$\left\{\text{"structure independent" IKMs}\right\} \leftrightarrow \left\{\text{S_∞-invariant measures with the appropriate language and age}\right\}$$

$$\left\{\text{invariant Keisler measures}\right\} \leftrightarrow \left\{\text{invariant random expansions in the sense of (Jahel \& Joseph 2023) with the appropriate language and age}\right\}$$

If $\mathrm{Age}(M)$ has the strong amalgamation property and disjoint 3-amalgamation, there is a homogeneous expansion $\mathcal{M}^*$ of $\mathcal{M}$ such that there is an IKM on $\mathcal{M}$ assigning positive measure to every non-forking formula if and only if there is an invariant random expansion of $\mathcal{M}$ concentrating on the isomorphism type of $\mathcal{M}^*$. ▸ More on this

Questions for the future:

Question 4

Are there invariant Keisler measures on the universal homogeneous 3-hypergraph which are NOT structure independent?

Question 5

On the generic tetrahedron-free 3-hypergraph $\mathcal{H}$, do we get that a formula forks over $\emptyset$ if and only if it is assigned measure zero by every invariant Keisler measures?

We can prove that "structure-independent" measures on $\mathcal{H}$ correspond to S_∞-invariant measures ν on Struc_E with $\mathrm{Age}(\nu) \subseteq \{$finite $\triangle$-free graphs$\}$. This already gives some non-trivial measures. But can we have

$$\mu(T(x,a,b) \wedge T(x,a,c) \wedge T(x,b,c)) > 0,$$

for abc distinct not forming a hyperedge?

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MS-measurable structures

Definition 21 (Macpherson & Steinhorn, 2008)

An infinite $\mathcal{L}$-structure is **MS-measurable** if there is a **dimension measure function** $h = (\dim, \mu) : \mathrm{Def}(M) \to \mathbb{N} \times \mathbb{R}^{>0}$ such that:

Finiteness $h(\phi(\overline{x}, \overline{a}))$ has finitely many values as $\overline{a} \in M^m$ varies;

Definability The set of $\overline{a} \in M^m$ such that $h(\phi(\overline{x}, \overline{a}))$ has a given value is $\emptyset$-definable;

Algebraicity For $|\phi(M^n, \overline{a})|$ finite, $h(\phi(\overline{x}, \overline{a})) = (0, |\phi(M^n, \overline{a})|)$;

Additivity For $X, Y \subset M^n$ definable and disjoint

$$\mu(X \cup Y) = \begin{cases} \mu(X) + \mu(Y), & \text{for } \dim(X) = \dim(Y); \\ \mu(X), & \text{for } \dim(Y) < \dim(X). \end{cases}$$

Fubini for Projections Let $X \subseteq M^n$ be definable, $\pi : M^n \to M$ be the projection on the i^{th} coordinate. Suppose for each $a \in \pi(X)$ $h(\pi^{-1}(a) \cap X) = (d, \nu)$. Then, $\dim(X) = \dim(\pi(X)) + d$ and $\mu(X) = \mu(\pi(X)) \times \nu$.

Basic facts about MS-measurable structures

Macpherson & Steinhorn (2008):

Remark 22

- Being MS-measurable is a property of a theory;
- MS-measurable structures are supersimple of finite SU-rank;
- If $\mathcal{M}$ is MS-measurable, then so is $\mathcal{M}^{eq}$.

Example 23

- Pseudofinite fields (Chatzidakis, Van den Dries & Macintyre, 1997);
- Random Graph (Macpherson & Steinhorn, 2008);
- ω-categorical ω-stable structures, and more generally smoothly approximable structures (Elwes 2005);

Go back to main presentation

Invariant Random Expansions

Definition 24 (Jahel & Joseph 2023)

Let $\mathcal{L} \subseteq \mathcal{L}^*$ be countable relational languages. Let $\mathcal{M}$ be a countable structure homogeneous in the language $\mathcal{L}$. Let $\mathrm{Struc}_{\mathcal{L}^*}(M)$ be the space of expansions $\mathcal{M}'$ of $\mathcal{M}$ to the language $\mathcal{L}^*$.

An **invariant random expansion** (IRE) of $\mathcal{M}$ to $\mathcal{L}^*$ is a Borel probability measure on $\mathrm{Struc}_{\mathcal{L}^*}(M)$ which is invariant under the action of $\mathrm{Aut}(M)$ on M.

At the moment we still have very few techniques to determine when an IRE concentrates on a given isomorphism type.

Projection languages and structures

Definition 25 (Projection language)

Expand $\mathcal{L}$ to the language $\mathcal{L}^* := \mathcal{L} \cup \mathcal{L}^p$ as follows: for each relation R_i of arity r_i and $K \subsetneq [r_i]$, $\mathcal{L}^p$ has a relation R_i^K of arity $r_i - |K|$.

Definition 26

Let $\mathcal{M}$ be an ω-categorical homogeneous $\mathcal{L}$-structure with $\mathrm{Aut}(M)$ acting transitively on M. For $Ab \in \mathrm{Age}(M)$ on $[n+1]$, where A is on $[n]$ and b corresponds to $n+1$ we define the $\mathcal{L}^*$-structure A^b. For $R_i \in \mathcal{L}$, and $m_1, \ldots, m_{r_i} \leq n$,

$$A^b \vDash R_i(m_1, \ldots, m_{r_i}) \text{ if and only if } A \vDash R_i(m_1, \ldots, m_{r_i}).$$

For $K \subsetneq [r_i]$ and $m_1, \ldots, m_{r_i - |K|} \leq n$, let $\overline{m} = (m_1, \ldots, m_{r_i - |K|})$ and $\overline{m}^K$ be the r_i-tuple consisting of $'n+1'$ in each position in K and the m_j in the other positions. Hence, we set that

$$A^b \vDash R_i^K(\overline{m}) \text{ if and only if } Ab \vDash R_i(\overline{m}^K).$$

More correspondences

Definition 27

We define the **measuring class** of $\mathcal{M}$, $\mathrm{MC}(\mathcal{M})$ as the class of finite $\mathcal{L}^*$-structures isomorphic to some A^b for $Ab \in \mathrm{Age}(M)$.

Theorem 28

Let $\mathcal{M}$ be ω-categorical and homogeneous in the language $\mathcal{L}$, with $\mathrm{Aut}(M)$ acting transitively on M. There is a one-to-one correspondence between invariant Keisler measures on $\mathcal{M}$ and invariant random expansions on of $\mathcal{M}$ to $\mathcal{L}^*$ with age contained in $\mathrm{MC}(\mathcal{M})$.

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