EPPA numbers of graphs

Matěj Konečný

~~Charles University~~ $\longrightarrow$ TU Dresden

$\mathrm{G^2OAT}$ Monday seminar 2023

David Bradley-Williams, Peter J. Cameron, Jan Hubička, and MK: EPPA numbers of graphs (arXiv:2311.07995)

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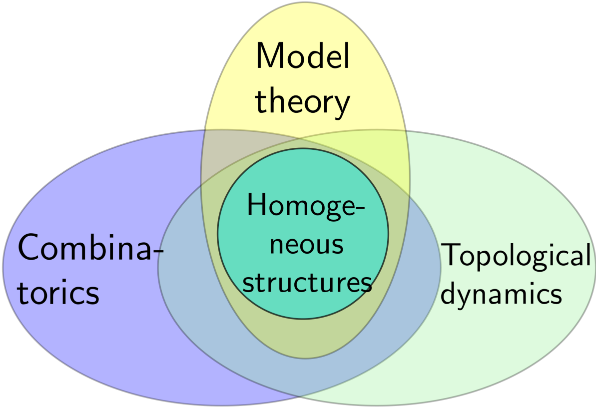
Model theory
Combinatorics
Homogeneous structures
Topological dynamics

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Example

A graph **G** is vertex-transitive if every partial automorphism f with $|\mathrm{Dom}(f)| \leq 1$ extends to an automorphism of **G**.

Definition (EPPA, extension property for partial automorphisms)

Let **B** be a structure and let **A** be its **induced** substructure. **B** is an EPPA-witness for **A** if every partial automorphism of **A** extends to an automorphism of **B**.

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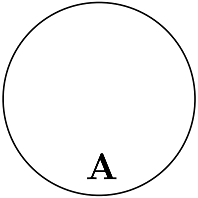

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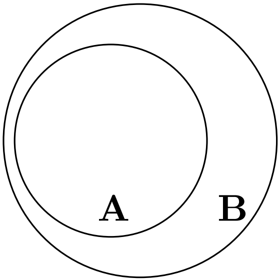

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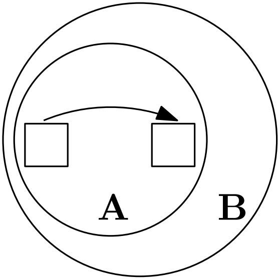

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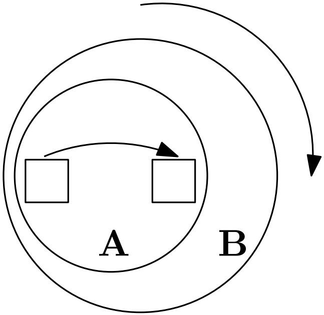

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Theorem (Hrushovski, 1992)

The class of all finite graphs has EPPA.

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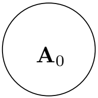

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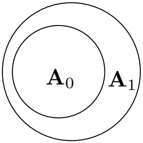

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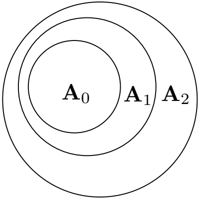

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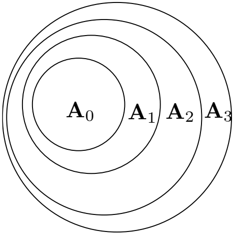

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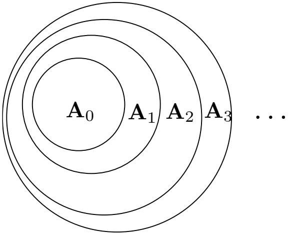

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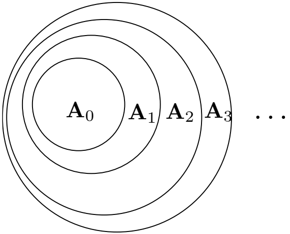

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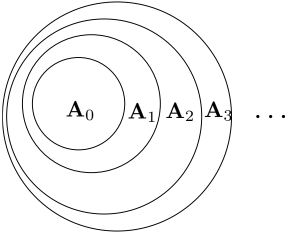

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Theorem [Kechris, Rosendal, 2007]: *The class of all finite substructures of a homogeneous structure* **M** *has EPPA if and only if* $\mathrm{Aut}(\mathbf{M})$ *can be written as the closure of a chain of compact subgroups.*

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- complements thereof,
- subgraphs of the finite homogeneous graphs [Gardiner, 1976].

Examples

- Graphs [Hrushovski, 1992], K_n-free graphs [Herwig, 1998]
- Relational structures (with forbidden cliques) [Herwig, 1995], [Hodkinson, Otto, 2003]
- Metric spaces [Solecki, 2005; Vershik, 2008], also [Conant, 2019]
- Two-graphs [Evans, Hubička, K, Nešetřil, 2018]
- Metrically homogeneous graphs [AB-WHKKKP, 2017], [K, 2019]
- Generalised metric spaces [Hubička, K, Nešetřil, 2019+]
- n-partite and semigeneric tournaments [Hubička, Jahel, K, Sabok, 2019+]
- Groups [Siniora, 2017]
- ...

Question (Herwig, Lascar, 2000)

Do finite tournaments have EPPA?

EPPA numbers of graphs

Given a graph $\mathbf{G}$, let $\mathrm{eppa}(\mathbf{G})$ be the least number of vertices of an EPPA-witness for $\mathbf{G}$. Put $\mathrm{eppa}(n) = \max\{\mathrm{eppa}(\mathbf{G}) : |\mathbf{G}| = n\}$.

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Problem (Hrushovski, 1992)

Improve the bounds.

Theorem (Herwig, Lascar, 2000)

For every $\mathbf{G}$ *with* n *vertices and maximum degree* Δ *we have that* $\mathrm{eppa}(\mathbf{G}) \leq \binom{\Delta n}{\Delta} \in n^{\mathcal{O}(n)}$.
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Theorem (Evans, Hubička, K, Nešetřil, 2021)

$$\mathrm{eppa}(n) \leq n2^{n-1}$$

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If the maximum degree of $\mathbf{G}$ *is* Δ*, then it has an EPPA-witness on at most* $\binom{\Delta n}{\Delta}$ *vertices.*

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Proof.

1. Let $\mathbf{G} = (V, E)$ be a graph. Assume that $\mathbf{G}$ is Δ-regular.
2. Define $\mathbf{H}$ so that $V(\mathbf{H}) = \binom{E}{\Delta}$ and $XY \in E(\mathbf{H})$ if $X \cap Y \neq \emptyset$.
3. Embed $\psi\colon \mathbf{G} \to \mathbf{H}$ sending $v \mapsto \{e \in E : v \in e\}$.
4. A partial automorphism of $\mathbf{G}$ gives a partial permutation of E.
5. Extend it to a permutation of E respecting the partial automorphism.
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For non-regular graphs, add "half-edges" to make them regular.

An upper bound [Evans, Hubička, K, Nešetřil, 2021]

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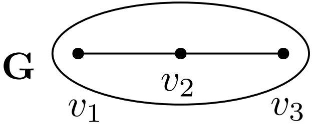

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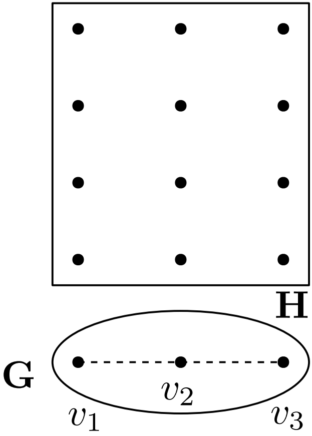

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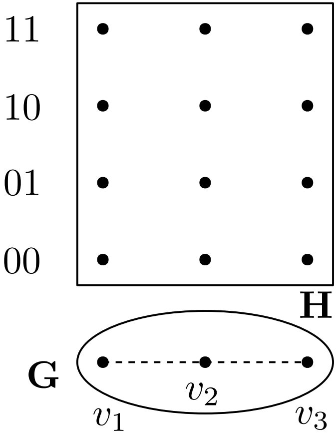

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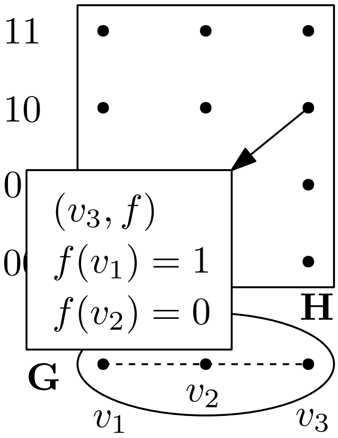

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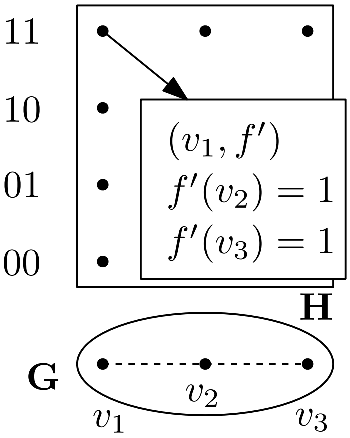

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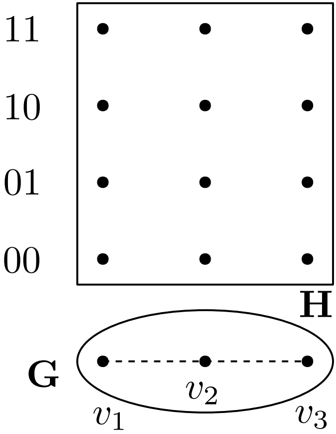

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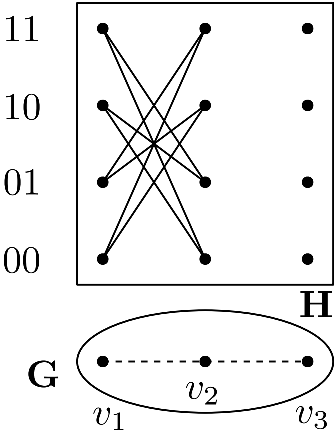

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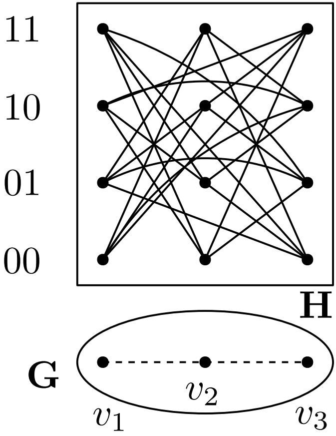

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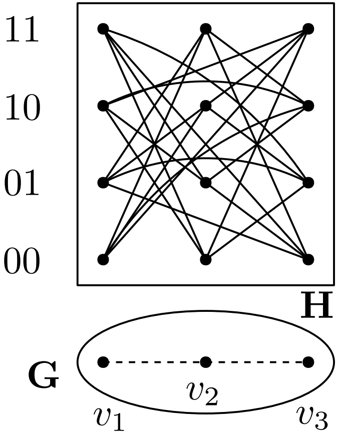

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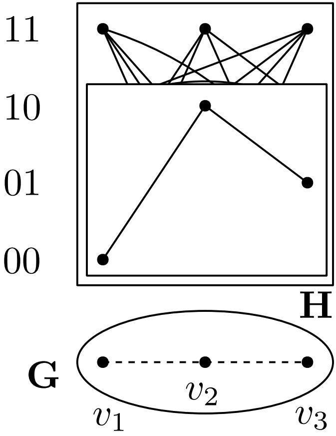

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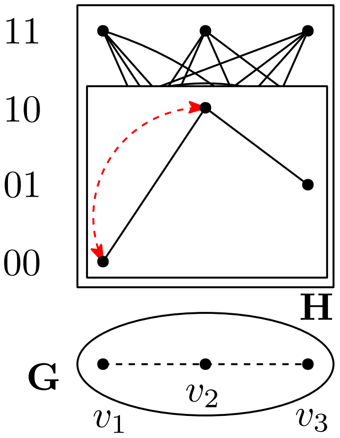

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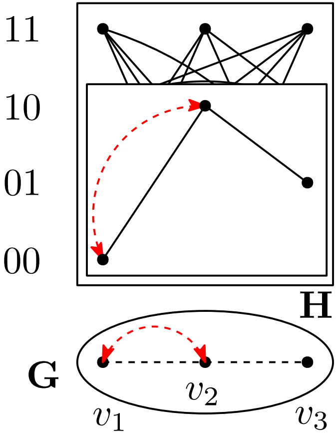

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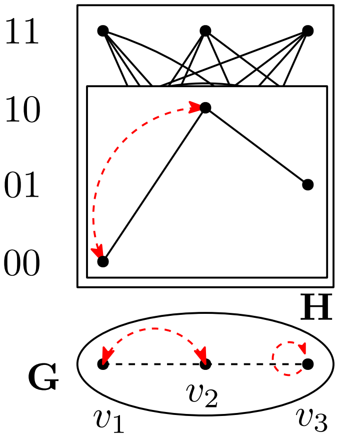

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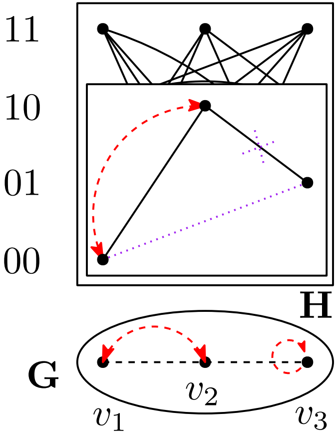

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- $\{(i, f), (i', f')\} \in E(\mathbf{H})$, if and only if $i \neq i'$ and $f(i') \neq f'(i)$.
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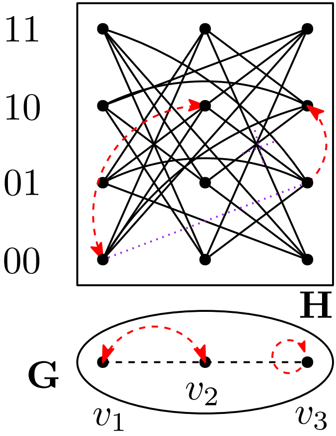

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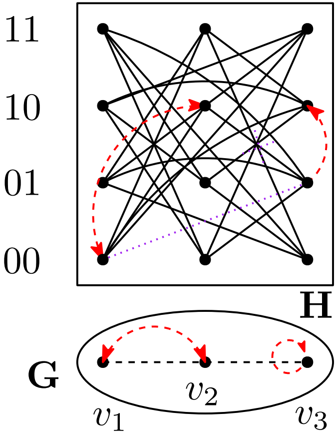

For $u, v \in G$, we define a flip $F_{u,v}((w, f)) = (w, f')$, where

$$f'(x) = \begin{cases} 1 - f(x) & \text{if } \{w, x\} = \{u, v\} \\ f(x) & \text{otherwise.} \end{cases}$$

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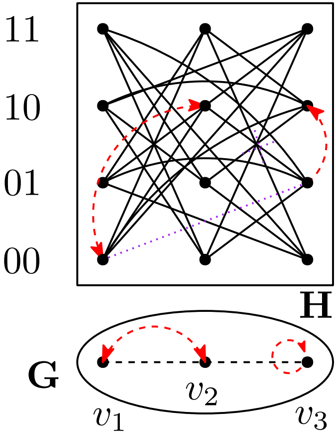

Remark

This can be straightforwardly generalised to hypergraphs and arbitrary relational structures, and one can also add unary functions.

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Observation (Bradley-Williams, Cameron, Hubička, Konečný, 2023)

There is **G** *such that every EPPA-witness for* **G** *has at least* $\Omega(2^n/\sqrt{n})$ *vertices. Consequently,* $\mathrm{eppa}(n) \geq \Omega(2^n/\sqrt{n})$.

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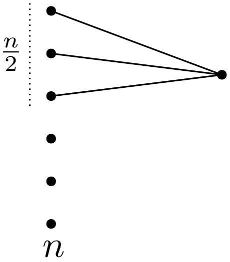

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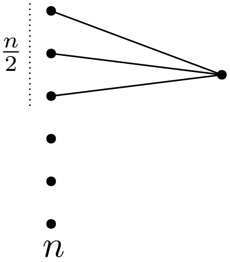

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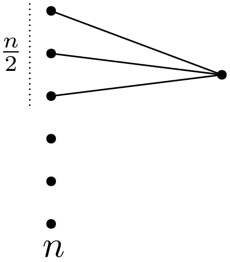

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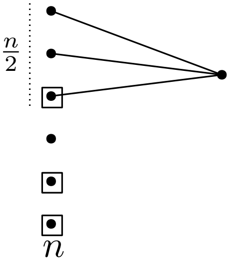

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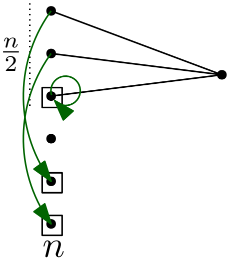

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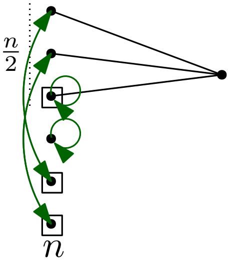

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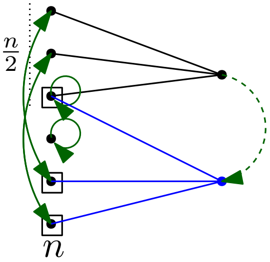

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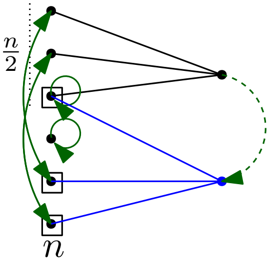

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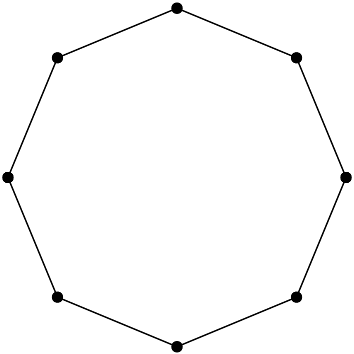

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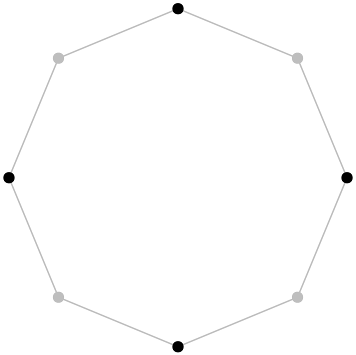

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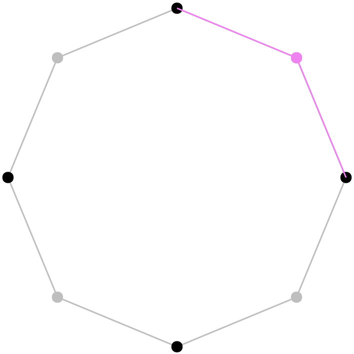

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For every m, *there is a 3-uniform hypergraph* $\mathbf{G}$ *on* $n = 2^m + m + 1$ *vertices with* $\mathrm{eppa}_3(\mathbf{G}) \geq m! \in 2^{\Omega(n \log n)}$.

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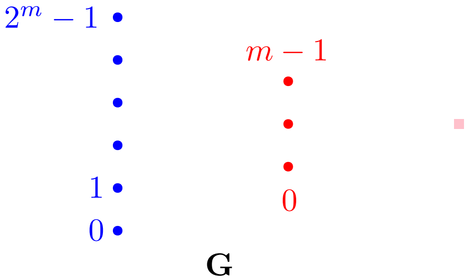

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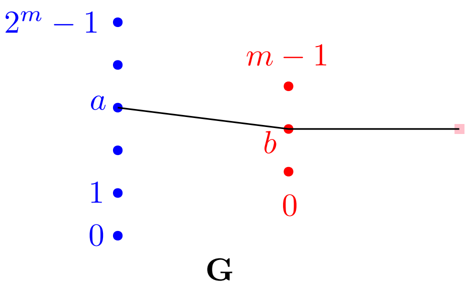

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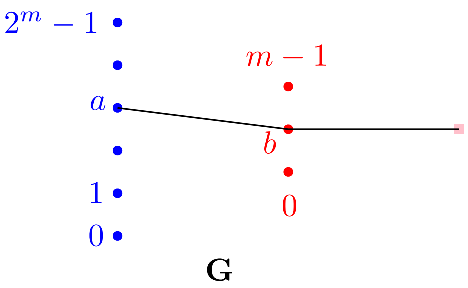

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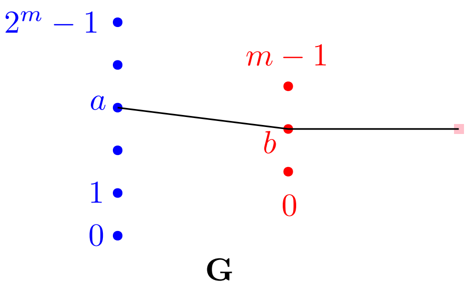

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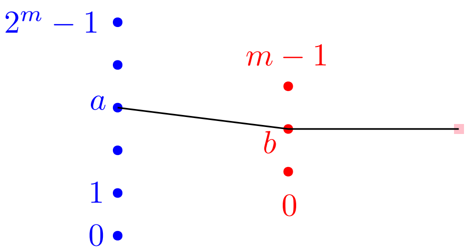

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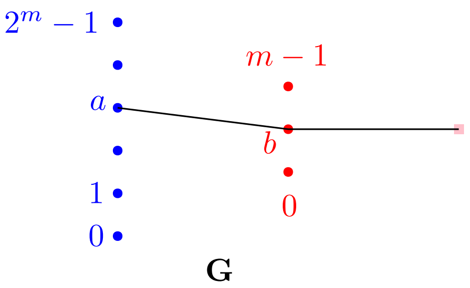

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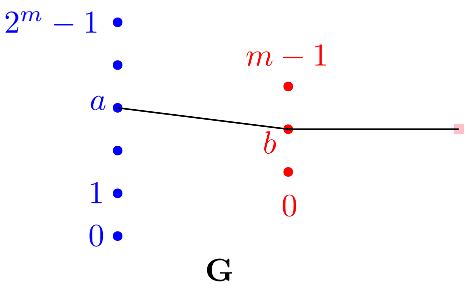

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- Note that there are only $2^{\mathcal{O}(n \log n)}$ partial permutations.

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Close the gap $\Omega(2^n/\sqrt{n}) \leq \mathrm{eppa}(n) \leq n2^{n-1}$. (I doubt the lower bound is tight.)

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(Answers?)

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