Valued Constraint Satisfaction Problems and Resilience in Database Theory

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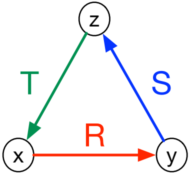
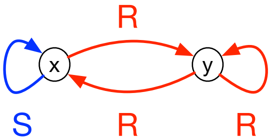
Ho!

Overview



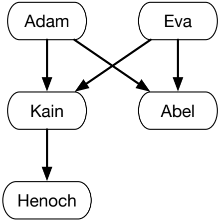


1. Resilience in Database Theory
2. Complexity of Resilience
3. Connection with Valued Constraint Satisfaction Problems
4. Universal-Algebraic Approach
5. NP-hardness and polynomial-time tractability
6. Tractability Conjecture

Conjunctive Queries

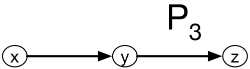
Database: relational structure $\mathfrak{A}$.

x is parent	of y
Adam	Kain
Eva	Kain
Adam	Abel
Eva	Abel
Kain	Henoch



Conjunctive query: primitive positive formula q, e.g.

$$\exists x, y, z \big(\text{parent}(x, y) \wedge \text{parent}(y, z)\big)$$



In our example:

$$\mathfrak{A} \models q$$

$$P_3 \to \mathfrak{A}$$

Resilience

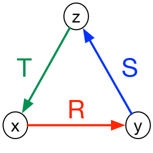
Resilience problem: How many tuples must be removed from relations of $\mathfrak{A}$ s.t.

$$\mathfrak{A} \not\models q?$$

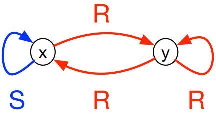
Computational complexity depends on q!

Examples. Meliou+Gatterbauer+Moore+Suciu (DVLDB'10), Freire+Gatterbauer+Immerman+Meliou (VLDB'2015,PODS'20).

- $\exists x, y, z(R(x,y) \wedge S(y,z) \wedge T(z,x))$.
  Resilience problem is NP-hard.



- $\exists x, y(R(x,y) \wedge R(y,y) \wedge R(y,x) \wedge S(x))$
  Complexity left open in PODS'20.



Research Goal:
Classify complexity of resilience for all conjunctive queries q!

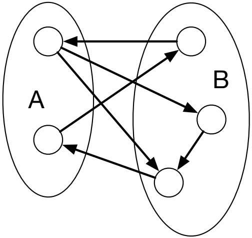
Valued Constraint Satisfaction Problems

Given: a finite set of variables, a finite set of constraints.

- **CSP (Constraint Satisfaction Problem):**
  decide whether there exists a solution that satisfies all constraints.
- **Max CSP:** find a solution that satisfies as many constraints as possible.
- **Valued CSP:** Find solution of minimal cost: each constraint comes with costs depending on the chosen values.

Example. Max Cut (NP-hard)
Given a finite directed graph (V, E), find a partition A, B of V such that

- $E \cap (A \times B)$ is maximal.
- Equivalently: $E \cap (A^2 \cup B^2 \cup B \times A)$ is minimal.



Valued Structures

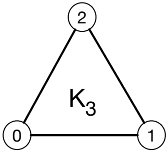
Γ: valued structure.
(Countable) domain D.
(Finite, relational) signature τ.
For each $R \in \tau$ of arity k, function $R^\Gamma \colon D^k \to \underbrace{\mathbb{Q} \cup \{\infty\}}_{\text{'costs'}}$.

Example 1. Γ_{MC}.
$D = \{0, 1\}$.
$\tau = \{E\}$ where E is binary relation symbol.
$E^{\Gamma_{MC}} \colon D^2 \to \mathbb{Q} \cup \{\infty\}$ given by

$$E^{\Gamma_{MC}}(a, b) = \begin{cases} 0 & \text{if } a = 0 \text{ and } b = 1, \\ 1 & \text{otherwise.} \end{cases}$$

Example 2. K_3. $D = \{0, 1, 2\}$, $\tau = \{E\}$.

$$E^{K_3}(a, b) = \begin{cases} 0 & \text{if } a \neq b, \\ \infty & \text{otherwise.} \end{cases}$$



VCSPs, Formal Definition

Fixed: Γ.

Definition (VCSP(Γ))

Input: $u \in \mathbb{Q}$, and an expression ϕ of the form

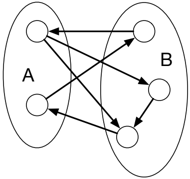
$$\inf_{x \in D^n} \sum_{i \in \{1,\dots,m\}} \psi_i$$

where each ψ_i is of the form $R(x_{i_1}, \dots, x_{i_k})$
for $R \in \tau$ of arity k and $i_1, \dots, i_k \in \{1, \dots, n\}$.

Question: $\phi \leq u$ in Γ?

Examples.

- VCSP(Γ_{MC}) is the Max Cut Problem!
- VCSP(K_3) is 3-colorability Problem!



VCSP Dichotomy

Γ: valued structure with a finite domain.

Theorem.

VCSP(Γ) is in P or NP-hard.

Guide to the literature:

- Živný+Thapper (STOC'13): proof if no ∞ costs.
- Kozik+Ochremiak (ICALP'15): hardness condition.
  If hardness condition does not apply:
  Γ has cyclic fractional polymorphism of arity at least two.
- Kolmogorov+Rolínek+Krokhin (FOCS'15): in this case, VCSP(Γ) is in P if the finite-domain Feder-Vardi CSP dichotomy conjecture is true.
- Bulatov (FOCS'17), Zhuk (FOCS'17):
  proof of Feder-Vardi conjecture.

Resilience Problems as VCSPs

Homomorphism duality: for every finite digraph G we have

$$P_3 \not\to G \text{ if and only if } G \to P_2$$

Turn P_2 into a valued structure Γ with signature $\{E\}$: define

$$E^\Gamma(a, b) := \begin{cases} 0 & \text{if } (a, b) \in E \\ 1 & \text{otherwise} \end{cases}$$

Note: $\Gamma = \Gamma_{MC}$!
Consequence: The following problems are identical:

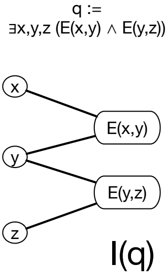
- The resilience problem for $q := \exists x, y, z \big(R(x, y) \wedge R(y, z) \big)$ (the same tuple might appear multiple times in the database)
- The VCSP for Γ_{MC}.

Consequence: Resilience problem for q is NP-hard.

For which queries q is there a dual structure $\mathfrak{B}$ such that for every finite structure $\mathfrak{A}$

$$\mathfrak{A} \not\models q \text{ if and only if } \mathfrak{A} \to \mathfrak{B} \text{ ?}$$

Definition. incidence graph $I(q)$:
bipartite undirected multigraph.
First colour class: variables of q.
Second colour class: conjuncts of q.
Edges link conjuncts with their variables.



Theorem (Nešetřil+Tardiff'00; Larose+Loten+Tardif'07; Foniok'07).
A conjunctive query q has a finite dual if and only if $I(q)$ is a tree.

Consequences

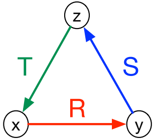
Theorem (B.+Lutz+Semanišinová).
Let q be a conjunctive query such that $I(q)$ is a tree.
Then the resilience problem for q is NP-hard or in P.

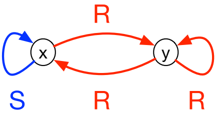
Proof idea: turn the finite dual $\mathfrak{B}_q$ of q into a valued structure Γ_q (all cost functions take values in $\{0, 1\}$).

Generalisations:

1. Presence of 'exogenous' tuples:
   the tuples for some specified relations may not be removed.
   use cost ∞ instead of 1 in the dual.
2. '(Finite) unions of conjunctive queries' instead of conjunctive queries.
3. It suffices that $I(q)$ is acyclic.

But what if $I(q)$ contains cycles?





Cherlin-Shelah-Shi

q: conjunctive query.

Theorem (Cherlin+Shelah+Shi Adv.Appl.Math'99).

If $I(q)$ is connected, then q has a countable dual $\mathfrak{B}$.
$\mathfrak{B}$ can be chosen so that $\mathrm{Aut}(\mathfrak{B})$ is oligomorphic.

A permutation group G on a countably infinite set B is called oligomorphic if $G \curvearrowright B^n$ has finitely many orbits for every $n \geq 1$.

Example. $\mathrm{Aut}(\mathbb{Q}; <)$ is oligomorphic.
(However, $(\mathbb{Q}; <)$ is not a dual of a single conjunctive query.)

Fact. If G is oligomorphic, then $|G| = 2^{\aleph_0}$.

Model-Complete Cores

q: conjunctive query.

Theorem (B.'06).

The dual $\mathfrak{B}$ of q can be chosen so that it is

- model complete: every first-order formula is in $\mathfrak{B}$ equivalent to an existential formula;
- a core: every homomorphism from $\mathfrak{B}$ to $\mathfrak{B}$ is an embedding.

Moreover, $\mathfrak{B}$ is up to isomorphism uniquely described by these properties, and $\text{Aut}(\mathfrak{B})$ is oligomorphic.

Example. Let $q = \exists x, y \big(E(x, y) \wedge E(y, x) \big)$.
Then $\mathfrak{B}$ is the so-called random tournament:
the up to isomorphic unique model of the almost-sure theory of the uniform distribution on finite tournaments of size n.

Consequences

q: conjunctive query such that $I(q)$ is connected.
$\mathfrak{B}_q$: model-complete core dual of q.
Γ_q: valued structure obtained from $\mathfrak{B}_q$.

Theorem (B., Lutz, Semanišinová).
The resilience problem for q equals $\mathrm{VCSP}(\Gamma_q)$.

Again:

- Also works with exogeneous tuples.
- Also works for unions of conjunctive queries.
- Assumption that $I(q)$ is connected can be made wlog.

Expressive Power of Valued Structures

Γ: valued structure with domain D and signature τ.
ϕ: τ-expression $\sum_{i \in \{1,\dots,m\}} \psi_i$.
$R \colon D^k \to \mathbb{Q} \cup \infty$.

Definition. $\phi(x_1, \dots, x_k, y_1, \dots, y_l)$ expresses R in Γ if for all $a \in D^k$

$$R(a) = \inf_{b \in D^k} \phi^\Gamma(a, b)$$

Fact. If $\mathrm{Aut}(\Gamma)$ is oligomorphic, then $\mathrm{VCSP}(\Gamma, R)$ reduces to $\mathrm{VCSP}(\Gamma)$.

Other complexity-preserving expansions of Γ:

- $R_\emptyset(a) := \infty$ for all $a \in D$.
- $R_=(a, b) := 0$ if $x = y$ and $R_=(a, b) = \infty$ otherwise.
- non-negative scaling: $r \cdot R$ for $r \in \mathbb{Q}_{\geq 0}$.
- shifting: $R + s$ for $s \in \mathbb{Q}$.
- $\mathrm{Feas}(R) := \{a \in D^k \mid R(a) < \infty\}$.
- $\mathrm{Opt}(R) := \{a \in \mathrm{Feas}(R) \mid R(a) \leq R(b) \text{ for every } b \in D^k\}$.

Hardness

Definition

- $\langle\Gamma\rangle$: valued structure obtained from Γ by adding $R_\emptyset$ and $R_=$ and closing under expressibility, non-negative scaling, shifting, Feas, and Opt.
- d-th pp-pwer of Γ: valued structure Δ with domain D^d such that for every R of arity k in Δ there exists S of arity dk in $\langle\Gamma\rangle$ such that

$$R\big((a_1^1,\dots,a_d^1),\dots,(a_1^k,\dots,a_d^k)\big) = S(a_1^1,\dots,a_d^1,\dots,a_1^k,\dots,a_d^k).$$

- Γ pp-constructs Δ if Δ is fractionally homomorphically equivalent to a pp-power of Γ.

Fact. If $\mathrm{Aut}(\Gamma)$ is oligomorphic and Γ pp-constructs Δ, then $\mathrm{VCSP}(\Delta)$ reduces to $\mathrm{VCSP}(\Gamma)$.

Corollary. If $\mathrm{Aut}(\Gamma)$ is oligomorphic and Γ pp-constructs K_3, then $\mathrm{VCSP}(\Gamma)$ is NP-hard.

Fractional Homomorphisms

Definition. A fractional map from D to C is a probability distribution

$$\big(C^D, \underbrace{\mathcal{B}(C^D)}_{\text{Borel } \sigma\text{-algebra}}, \omega\colon \mathcal{B}(C^D) \to [0,1]\big).$$

A fractional homomorphism between valued structures Δ to Γ with the same signature τ and domains D and C is a fractional map from D to C such that for every $R \in \tau$ of arity k and every $a \in D^k$

$$E_\omega[f \mapsto R^\Gamma(f(a))]$$

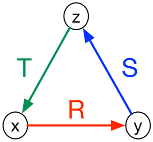
exists (always exists if $\mathrm{Aut}(\Gamma)$ is oligomorphic) and

$$E_\omega[f \mapsto R^\Gamma(f(a))] \leq R^\Delta(a).$$

Remarks.

- Fractional homomorphisms compose.
- Hence: may define fractional homomorphic equivalence.
- Fractional homomorphic equivalence preserves complexity of VCSP.

Example



$q := \exists x, y, z (R(x, y) \wedge S(y, z) \wedge T(z, x))$

Claim. Γ_q pp-constructs Γ_{K_3}.

Consequences.

- $\mathrm{VCSP}(\Gamma_q)$ is NP-hard.
- Resilience problem for q is NP-hard.

Fractional Polymorphisms

Γ: valued structure with domain D and signature τ.
Fractional polymorphism of Γ:
fractional homomorphism ω from specific pp power Γ^ℓ to Γ:
for every $R \in \tau$ of arity k

$$R^{\Gamma^\ell}((a^1_1, \dots, a^1_\ell), \dots, (a^k_1, \dots, a^k_\ell)) := \frac{1}{\ell} \sum_{i \in \{1,\dots,\ell\}} R^\Gamma(a^1_i, \dots, a^k_i).$$

Idea. Expected cost of a k-tuple obtained from applying ω to ℓ tuples is at most the average cost of these tuples.

Example. $\pi^\ell_i \colon D^\ell \to D$ given by $\pi^\ell_i(x_1, \dots, x_\ell) = x_i$.
Id_ℓ given by $\mathrm{Id}_\ell(\{\pi^\ell_i\}) := \frac{1}{\ell}$ for every $i \in \{1, \dots, \ell\}$
is fractional polymorphism for every Γ.

Polynomial-time Tractability

$f\colon D^\ell \to D$ is cyclic if for all $x_1, \dots, x_\ell \in D$:

$$f(x_1, \dots, x_\ell) = f(x_2, \dots, x_\ell, x_1).$$

ω is called cyclic if for every $A \in \mathcal{B}(D^{D^\ell})$ we have

$$\omega(A) = \omega(\{f \in A \mid f \text{ is cyclic}\})$$

Theorem.

Γ: valued structure over finite domain. Then

- If K_3 has no pp-construction in Γ, then Γ has cyclic fractional polymorphism of arity $\ell \geq 2$ (essentially Kozik+Ochremiak).
- If Γ has cyclic fractional polymorphism of arity $\ell \geq 2$, then VCSP(Γ) is in P (Kolmogorov+Krokhin+Rolínek)

Tractability Conjecture

q: conjunctive query.

Conjecture. If K_3 does not have a pp-construction in Γ_q, then

- $\mathrm{VCSP}(\Gamma_q)$ is in P and
- the resilience problem for q is in P.

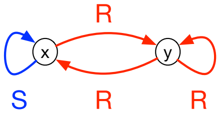
Theorem (B.,Lutz,Semanišinová).

If Γ_q has fractional polymorphism which is canonical and pseudo-cyclic with respect to $\mathrm{Aut}(\Gamma_q)$, then $\mathrm{VCSP}(\Gamma_q)$ is in P.

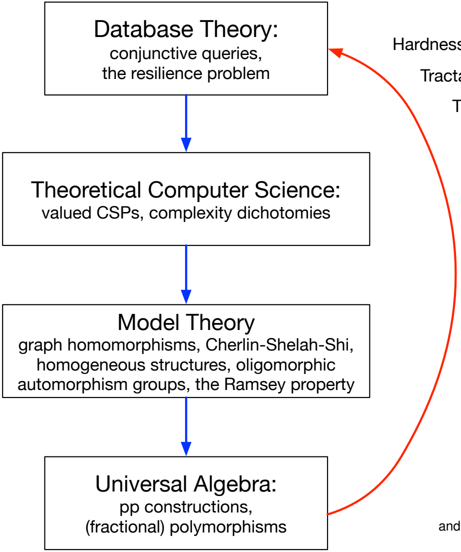
Proof by reduction to the finite, similarly as in B.+Mottet (LICS'16).

Example $\exists x, y \big(R(x,y) \wedge R(y,y) \wedge R(y,x) \wedge S(x)\big)$
Complexity left open at PODS'20.

Has such a polymorphism.



Summary



Hardness Conditions

Tractability Conditions

Tractability Conjecture

Dichotomy for Acylic Conjunctive Queries

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