Homogeneous graphs: construction, examples, and applications

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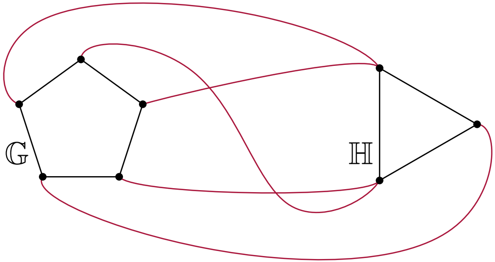
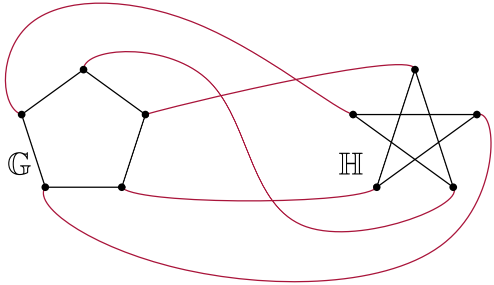
Graph homomorphism



Definition

For graphs $\mathbb{G}$ and $\mathbb{H}$, a mapping $f\colon V(\mathbb{G}) \to V(\mathbb{H})$ is a *homomorphism* if, for all $x, y \in V(\mathbb{G})$, xy is an edge in $\mathbb{G} \Rightarrow f(x)f(y)$ is an edge in $\mathbb{H}$.

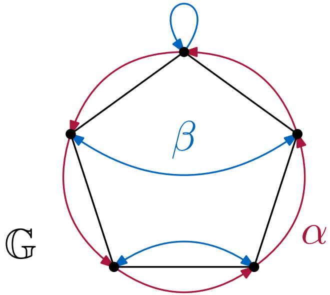
Graph isomorphism



Definition

For graphs $\mathbb{G}$ and $\mathbb{H}$, a bijective mapping $f\colon V(\mathbb{G}) \to V(\mathbb{H})$ is an *isomorphism* if both f and f^{-1} are homomorphisms.

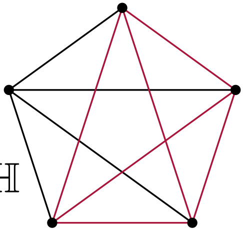
Graph automorphism



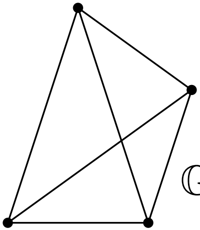
Definition

An isomorphism $a\colon \mathbb{G} \to \mathbb{G}$ from a graph $\mathbb{G}$ to itself is an *automorphism*. The automorphisms of $\mathbb{G}$ form a group denoted $\mathrm{Aut}(\mathbb{G})$.

Induced subgraph



$\mathbb{H}$

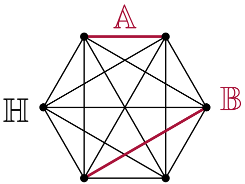


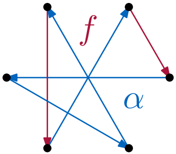
$\mathbb{G}$

Definition

- A graph $\mathbb{G}$ is an *induced subgraph* of a graph $\mathbb{H}$ if $V(\mathbb{G}) \subseteq V(\mathbb{H})$ and $E(\mathbb{G}) = E(\mathbb{H}) \cap \big(V(\mathbb{G})\big)^2$.
- Let $\mathrm{Age}(\mathbb{H})$ denote the set of all induced subgraphs of $\mathbb{H}$ up to isomorphism.
- If $\mathbb{G}$ is isomorphic to an induced subgraph of $\mathbb{H}$, then $\mathbb{G}$ *embeds* into $\mathbb{H}$.

Homogeneous graph

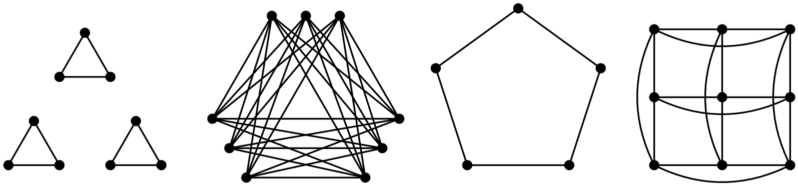




Definition

A graph $\mathbb{H}$ is *homogeneous* if it has countably many vertices and, for any finite induced subgraphs $\mathbb{A}, \mathbb{B}$ of $\mathbb{H}$ and any isomorphism $f\colon \mathbb{A} \to \mathbb{B}$, there exists $\alpha \in \mathrm{Aut}(\mathbb{H})$ such that $\alpha\big|_{V(\mathbb{A})} = f$.

Example: Finite homogeneous graphs

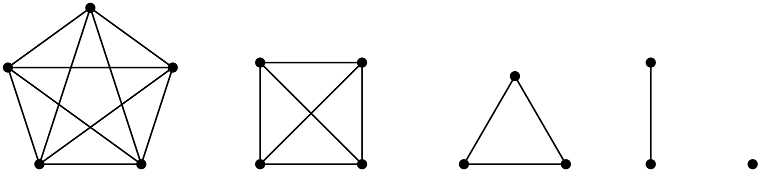


Theorem ([Gardiner, Golfand & Klin])

Let $\mathbb{G}$ be a finite homogeneous graph. Then either $\mathbb{G}$ or $\overline{\mathbb{G}}$ is isomorphic to a disjoint union of complete graphs all of the same size, or to the pentagon, or to the 3×3 rook's graph.

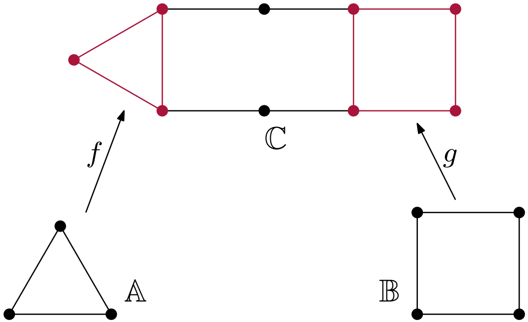
Construction of Infinite Graphs

Countably many finite graphs



The infinite homogeneous graph is constructed from a countable class $\mathcal{C}$ of finite graphs which is assumed to be closed under taking *isomorphisms*, taking *induced subgraphs*, and to have the *two following* properties.

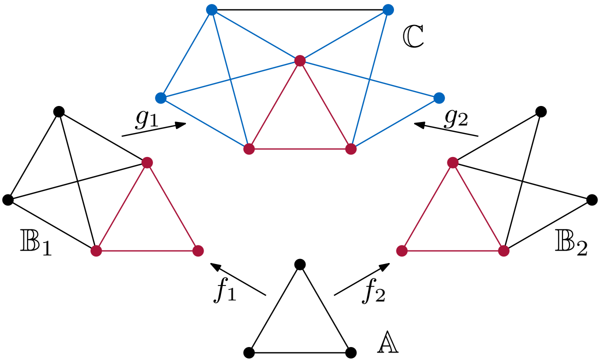
Joint Embedding Property (JEP)



Definition

A class $\mathcal{C}$ has the *joint embedding* property if, for any $\mathbb{A}, \mathbb{B} \in \mathcal{C}$, there exists $\mathbb{C} \in \mathcal{C}$ and embeddings $f\colon \mathbb{A} \to \mathbb{C}$ and $g\colon \mathbb{B} \to \mathbb{C}$.

Amalgamation Property (AP)



Definition

$\mathcal{C}$ has the *amalgamation* property if, for any $\mathbb{A}, \mathbb{B}_1, \mathbb{B}_2 \in \mathcal{C}$ and any embeddings $f_1\colon \mathbb{A} \to \mathbb{B}_1, f_2\colon \mathbb{A} \to \mathbb{B}_2$, there exists $\mathbb{C} \in \mathcal{C}$ and embeddings $g_1\colon \mathbb{B}_1 \to \mathbb{C}, g_2\colon \mathbb{B}_2 \to \mathbb{C}$ such that $g_1 \circ f_1 = g_2 \circ f_2$.

Fraïssé's Theorem

Theorem ([Fraïssé])

1. *Let* $\mathbb{M}$ *be a homogeneous graph. Then* $Age(\mathbb{M})$ *has the amalgamation property.*

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1. *Let $\mathbb{M}$ be a homogeneous graph. Then $Age(\mathbb{M})$ has the amalgamation property.*
2. *Let $\mathcal{C}$ be a non-empty class of finite graphs such that it is closed under taking isomorphisms and subgraphs, and that it has JEP and AP. Then there is a homogeneous graph $\mathbb{M}$ with $Age(\mathbb{M}) = \mathcal{C}$. $\mathbb{M}$ is called the Fraïssé limit of $\mathcal{C}$.*

Fraïssé's Theorem

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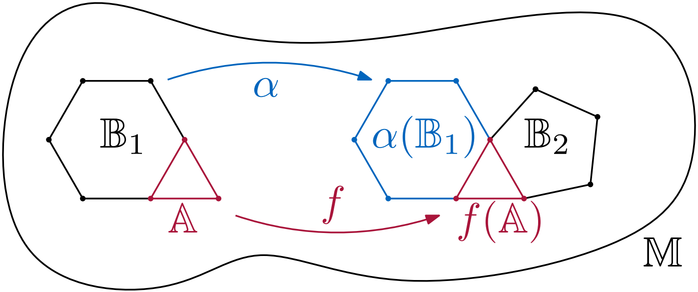
3. *Let $\mathbb{M}$ and $\mathbb{M}'$ be two homogeneous graphs such that $Age(\mathbb{M}) = Age(\mathbb{M}')$. Then $\mathbb{M}'$ is isomorphic to $\mathbb{M}'$.*

Proof of (1)



Without loss of generality, let $\mathbb{A}$ be an induced subgraph of $\mathbb{B}_1$. Let f be an embedding of $\mathbb{A}$ into $\mathbb{B}_2$, i.e. $\mathbb{A} \cong f(\mathbb{A})$. Then there is $\alpha \in \mathrm{Aut}(\mathbb{M})$ such that $\alpha\big|_{\mathbb{A}} = f$. The desired graph $\mathbb{C}$ is induced on the union of $\alpha(\mathbb{B}_1)$ and $\mathbb{B}_2$.

Proof of (2) – Construction

- Let $\theta\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be a bijection such that $\theta(i, j) \geq i$ for all $i, j \in \mathbb{N}$.

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- Construct the chain $\mathbb{M}_0 \subseteq \mathbb{M}_1 \subseteq \dots$ by induction and put $\mathbb{M} := \bigcup_{k=0}^{\infty} \mathbb{M}_k$.

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- Step 0. Take arbitrary $\mathbb{M}_0 \in \mathcal{C}$.
- Step $k+1$. $\mathbb{M}_k$ is associated with a countably infinite sequence $(\mathbb{A}_{kj}, \mathbb{B}_{kj}, f_{kj})_j$ of all triples $\mathbb{A}_{kj}, \mathbb{B}_{kj}, f_{kj}$ such that

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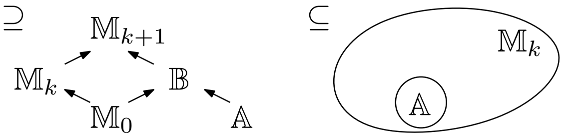
Proof of (2) – Checking the properties



$\mathrm{Age}(\mathbb{M}) = \mathcal{C}$

($\supseteq$) By JEP, for every $\mathbb{A}$ in $\mathcal{C}$ there is $\mathbb{B}$ in $\mathcal{C}$ such that both $\mathbb{M}_0$ and $\mathbb{A}$ embed into $\mathbb{B}$. The graph $\mathbb{B}$ is amalgamated to some $\mathbb{M}_k$ over $\mathbb{M}_0$, therefore $\mathbb{A}$ embeds into $\mathbb{M}_{k+1}$.

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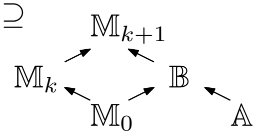

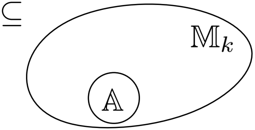

$\mathrm{Age}(\mathbb{M}) = \mathcal{C}$

($\supseteq$) By JEP, for every $\mathbb{A}$ in $\mathcal{C}$ there is $\mathbb{B}$ in $\mathcal{C}$ such that both $\mathbb{M}_0$ and $\mathbb{A}$ embed into $\mathbb{B}$. The graph $\mathbb{B}$ is amalgamated to some $\mathbb{M}_k$ over $\mathbb{M}_0$, therefore $\mathbb{A}$ embeds into $\mathbb{M}_{k+1}$.

($\subseteq$) Every $\mathbb{M}_k$ is a result of finitely many amalgamations, so it is in $\mathcal{C}$. Every finite $\mathbb{A} \subseteq \mathbb{M}$ is an induced subgraph of some $\mathbb{M}_k$. As $\mathcal{C}$ is closed under taking induced subgraphs, $\mathbb{A}$ is also in $\mathcal{C}$.

Proof of (2) – Checking the properties

The homogenity of $\mathbb{M}$ follows from this claim.

Claim

For all $\mathbb{A}, \mathbb{B} \in \mathcal{C}$ and for all embeddings $e\colon \mathbb{A} \to \mathbb{B}$ and $f\colon \mathbb{A} \to \mathbb{M}$, there is an embedding $g\colon \mathbb{B} \to \mathbb{M}$ such that $g \circ e = f$.

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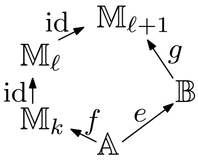
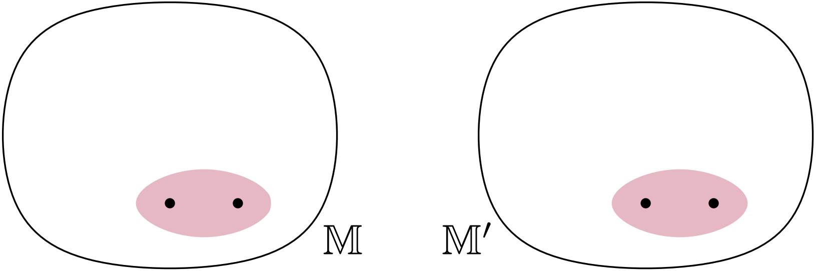


Proof.

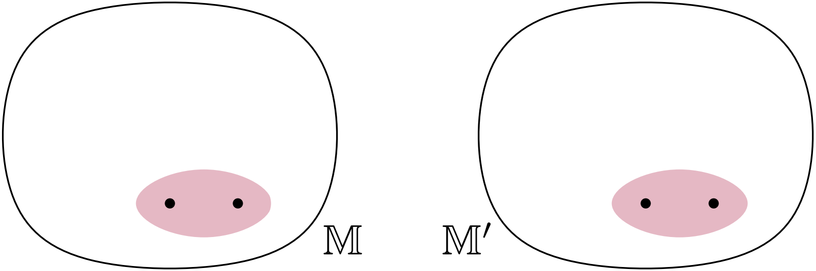
For some $k \in \mathbb{N}$, $\mathbb{A}$ embeds into $\mathbb{M}_k$. Then, the sequence $(\mathbb{A}_{kj}, \mathbb{B}_{kj}, f_{kj})_j$ contains the triple $(\mathbb{A}, \mathbb{B}, e)$. Then, for some $\ell \geq k$, $\mathbb{M}_{\ell+1}$ is the amalgamation of $\mathbb{M}_\ell$ and $\mathbb{B}$ over $\mathbb{A}$. □

Proof of (3) – Back-and-forth argument



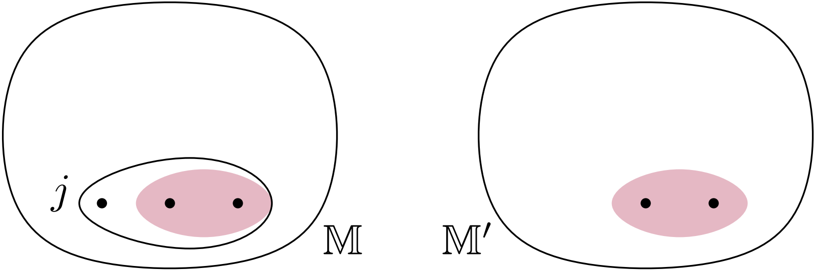
$\mathbb{M}$ and $\mathbb{M}'$ are homogeneous and $\mathrm{Age}(\mathbb{M}) = \mathrm{Age}(\mathbb{M}')$. Let $V(\mathbb{M}) = \{0, 1, \ldots\}$ and $V(\mathbb{M}') = \{0', 1', \ldots\}$.

Proof of (3) – Back-and-forth argument



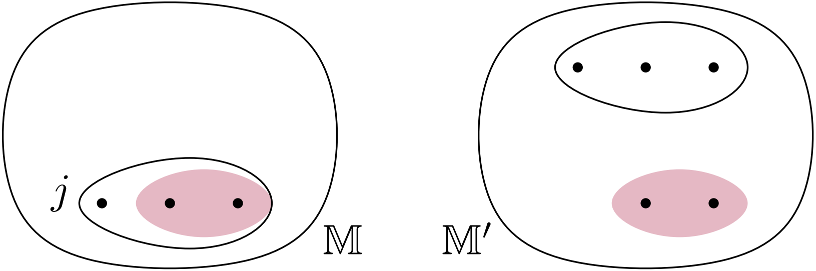
The isomorphism $f\colon \mathbb{M} \to \mathbb{M}'$ is constructed by induction.
Suppose that f is a partial isomorphism between $\mathbb{M}_i$ and $\mathbb{M}'_i$.

Proof of (3) – Back-and-forth argument



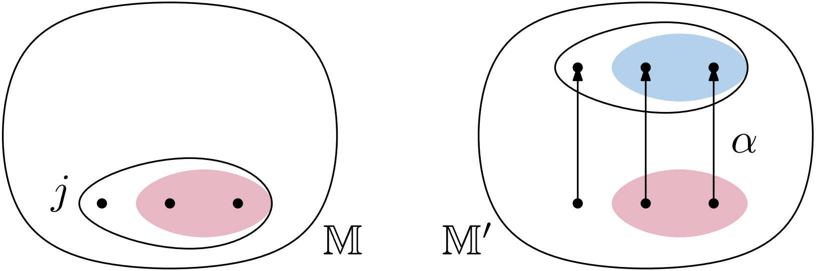
Take the least j in $\mathbb{M}$ which is not assigned to any element from $\mathbb{M}'$ and consider the subgraph induced on $\mathbb{M}_i \cup \{j\}$.

Proof of (3) – Back-and-forth argument



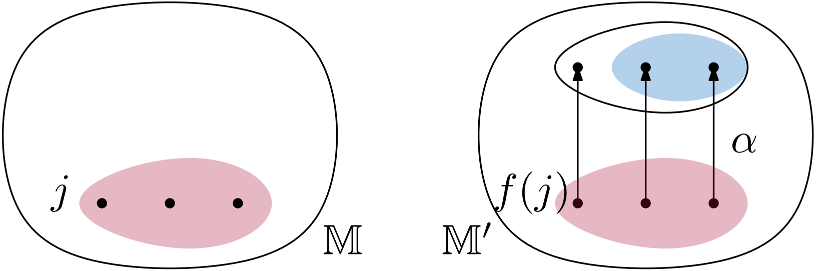
As $\mathrm{Age}(\mathbb{M}) = \mathrm{Age}(\mathbb{M}')$, the graph $\mathbb{M}'$ contains an induced subgraph isomorphic to $\mathbb{M}_i \cup \{j\}$.

Proof of (3) – Back-and-forth argument



As $\mathbb{M}'$ is homogeneous, there exists $\alpha \in \mathrm{Aut}(\mathbb{M}')$ that maps $\mathbb{M}'_i$ to this induced subgraph.

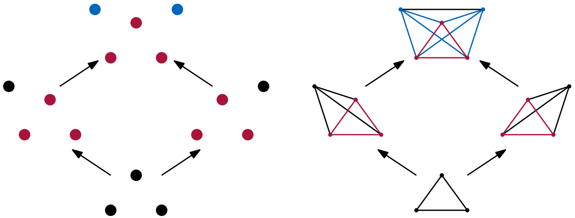
Proof of (3) – Back-and-forth argument



After assigning $f(j)$ to j, take the smallest unassigned j' in $\mathbb{M}'$ and similarly find the suitable $f^{-1}(j')$ in $\mathbb{M}$.

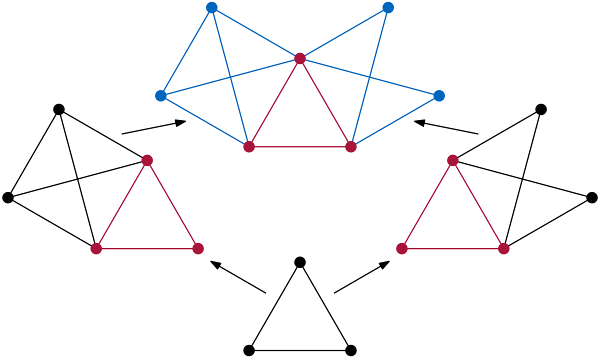
Examples

Independent sets and cliques



The class of finite independent sets and the class of all cliques both have AP. The Fraïssé limits are the countable independent set and the countable clique.

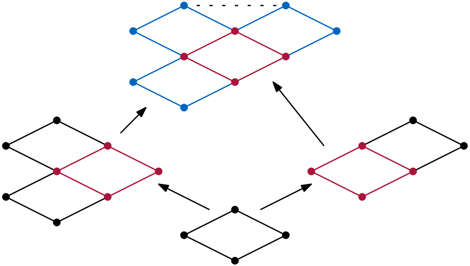
Random graph



The Fraïssé limit $\mathbb{M}$ of the class of all finite graphs is called *Rado* graph or *Erdős-Rényi* graph, or *random* graph. For $x, y \in V(\mathbb{M})$, xy is an edge with some fixed probability $p \in (0, 1)$.

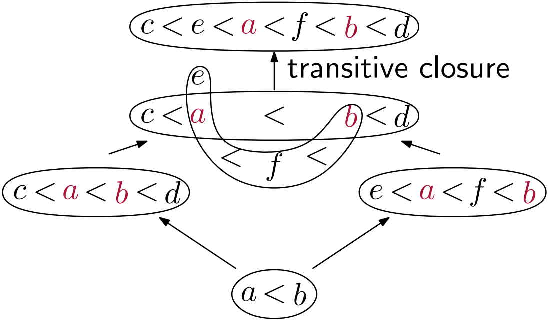
Universal clique-free graph

$\mathbb{K}_\ell =$ 



The class of graphs omitting $\mathbb{K}_\ell$ for some ℓ has AP, so there exists the universal homogeneous $\mathbb{K}_\ell$-free graph that is unique up to isomorphism.

Ordered sets



The classes of finite *partially* ordered sets and of finite *linearly* ordered sets both have AP. The Fraïssé limit of finite linear orders is isomorphic to $(\mathbb{Q}, <)$.

Constraint Satisfaction Problems

Constraint Satisfaction Problems

Definition

Let $\tau = \{R_1, \ldots, R_t\}$ be a relational signature. A τ-*structure* $\mathbb{A}$ is a tuple $\left(A; R_1^{\mathbb{A}}, \ldots, R_t^{\mathbb{A}}\right)$, where A is a set, and, for all $R \in \tau$ of arity k, we have that $R^{\mathbb{A}} \subseteq A^k$. The set A is the *domain* of $\mathbb{A}$ and each $R^{\mathbb{A}}$ is a *relation* of $\mathbb{A}$.

Constraint Satisfaction Problems

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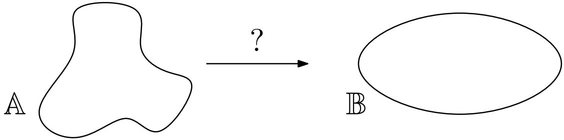
Let $\tau = \{R_1, \ldots, R_t\}$ be a relational signature. A *τ-structure* $\mathbb{A}$ is a tuple $\left(A; R_1^{\mathbb{A}}, \ldots, R_t^{\mathbb{A}}\right)$, where A is a set, and, for all $R \in \tau$ of arity k, we have that $R^{\mathbb{A}} \subseteq A^k$. The set A is the *domain* of $\mathbb{A}$ and each $R^{\mathbb{A}}$ is a *relation* of $\mathbb{A}$.



Definition

Let $\mathbb{B}$ be a τ-structure. The class $\mathrm{CSP}(\mathbb{B})$ contains all finite τ-structures $\mathbb{A}$ such that there is a homomorphism $\mathbb{A} \to \mathbb{B}$. The corresponding membership problem is also denoted $\mathrm{CSP}(\mathbb{B})$.

What is CSP?

Theorem

Let τ be a finite relational signature and let $\mathcal{C}$ be a class of finite τ-structures. Then the following are equivalent.

- $\mathcal{C} = CSP(\mathbb{B})$ *for a countable τ-structure $\mathbb{B}$.*

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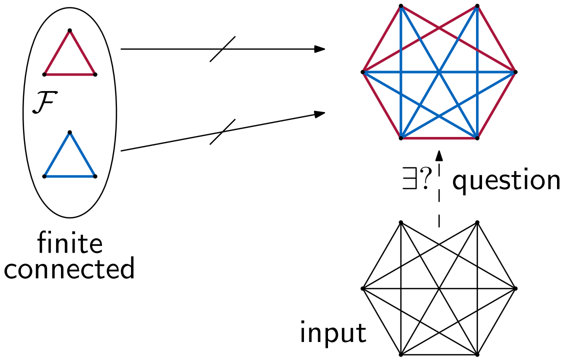
- $\mathcal{C} = CSP(\mathbb{B})$ *for a countable τ-structure $\mathbb{B}$.*
- $\mathcal{C}$ *is closed under disjoint unions and inverse homomorphisms.*

$$\mathbb{A}, \mathbb{B} \in \mathcal{C} \implies \mathbb{A} \sqcup \mathbb{B} \in \mathcal{C}$$

$$\mathbb{B} \in \mathcal{C} \text{ and } \mathbb{A} \to \mathbb{B} \implies \mathbb{A} \in \mathcal{C}$$

What do I study?



$\mathcal{F}$-free edge-coloring problems

How to color the input graph edges so that no edge-colored graph from $\mathcal{F}$ maps homomorphically to it?

Finitely boundedness

Definition

A family $\mathcal{C}$ of finite τ-structures is *finitely bounded* if there is a finite set of structures $\mathcal{F}$ such that, for any finite τ-structure $\mathbb{A}$, $\mathbb{A} \in \mathcal{C}$ iff no $\mathbb{F} \in \mathcal{F}$ embeds into $\mathbb{A}$, denoted $\mathcal{C} = \mathsf{Forb}_{\mathsf{emb}}(\mathcal{F})$.
A homogeneous structure $\mathbb{M}$ is *finitely bounded* if so is $\mathrm{Age}(\mathbb{M})$.

Finitely boundedness

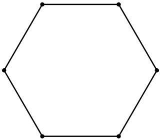
Definition

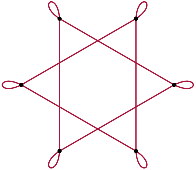
A family $\mathcal{C}$ of finite τ-structures is *finitely bounded* if there is a finite set of structures $\mathcal{F}$ such that, for any finite τ-structure $\mathbb{A}$, $\mathbb{A} \in \mathcal{C}$ iff no $\mathbb{F} \in \mathcal{F}$ embeds into $\mathbb{A}$, denoted $\mathcal{C} = \mathsf{Forb}_{\mathsf{emb}}(\mathcal{F})$.
A homogeneous structure $\mathbb{M}$ is *finitely bounded* if so is $\mathrm{Age}(\mathbb{M})$.

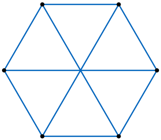
Question

Which homogeneous structures from "Examples" are finitely bounded?

First-order reducts of relational structures







Definition

A structure $\mathbb{B}$ is a *first-order reduct* of a structure $\mathbb{A}$ if they have the same set of vertices and if all relations of $\mathbb{B}$ are first-order definable from the relations of $\mathbb{A}$.

First-order reducts of finitely bounded homogeneous structures

Theorem ([CSS'1999, HN'2019, BMM'2021])

For every finite class $\mathcal{F}$ of connected finite structures there is a first-order reduct $\mathbb{M}$ of some homogeneous structure such that $Age(\mathbb{M}) = \textit{Forb}_{\mathsf{emb}}(\mathcal{F})$.

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Corollary

Every $\mathcal{F}$-free edge-coloring problem is exactly $CSP(\mathbb{B})$, where $\mathbb{B}$ is a first-order reduct of a finitely bounded homogeneous structure.

First-order reducts of finitely bounded homogeneous structures

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Corollary

Every $\mathcal{F}$-free edge-coloring problem is exactly $CSP(\mathbb{B})$, where $\mathbb{B}$ is a first-order reduct of a finitely bounded homogeneous structure.

Why?

If $\mathbb{B}$ is a first-order reduct of a finitely bounded homogeneous structure, then $\mathrm{CSP}(\mathbb{B})$ is in NP.

How can infinite CSPs help me?

	Dichotomy	Decidability of Containment
CSP(finite)	2017	decidable(obvious)
CSP(FORFBHS)	open	open
vertex-coloring	2017(1998)	decidable(2010)
edge-coloring	open	decidable (2023)(binary)

Definition

A family of NP-problems has a *dichotomy* if every its problem is either solvable in polynomial time or NP-hard.

Definition

A problem P_1 is *contained* in a problem P_2 if every finite input is accepted by P_1 only if it is accepted by P_2. For a class of problems $\mathcal{L}$, the containment is *decidable* if there is an algorithm running in finite time that checks for any given $P_1, P_2 \in \mathcal{L}$ whether $P_1 \subseteq P_2$.

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