Clones over finite sets and Minor Conditions

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Clones

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- $\mathcal{A}$ contains all projections;
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Example:

$P_2 := \langle \emptyset \rangle$ all projections over $\{0,1\}$;

$I_2 := \langle \wedge ; x +_2 y +_2 z \rangle$ all idempotent operations over $\{0,1\}$;

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Def: An operation $f: A^n \to A$ preserves a relation $R \subseteq A^k$ if $\forall\, a_1, \dots, a_n \in R$,
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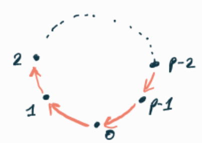
- An operation f is a polymorphism of $\mathbb{A} = (A; \Gamma)$ if f preserves R, for every $R \in \Gamma$.
- By $\mathrm{Pol}(\mathbb{A})$ we denote the set of all polymorphisms of $\mathbb{A}$. (it is a clone)

Example:

$C_p := \mathrm{Pol}(\{0, 1, \dots, p-1\}; \{(0,1), (1,2), \dots, (p-1, 0)\})$



$B_2 := \mathrm{Pol}(\{0,1\}; \{(0,1), (1,0), (1,1)\}, \{0\}, \{1\})$

A Galois Connection for Clones

On the relational side $\rightsquigarrow$ denote by $\mathrm{Inv}(F)$ the set $\{ R \mid f \text{ preserves } R, \forall f \in F \}$.

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Thm [Bodnarčuk, Kalužnin, Kotov, Romov; Geiger]: F: set of oper. over some finite universe; Γ: set of relations over some finite universe.

Then $\mathrm{Pol}(\mathrm{Inv}(F)) = \langle F \rangle$ and $\mathrm{Inv}(\mathrm{Pol}(\Gamma)) = [\Gamma]$ — relational clone generated by R

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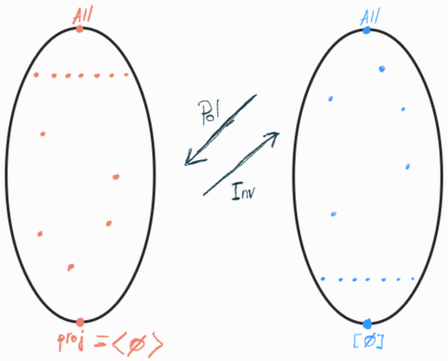
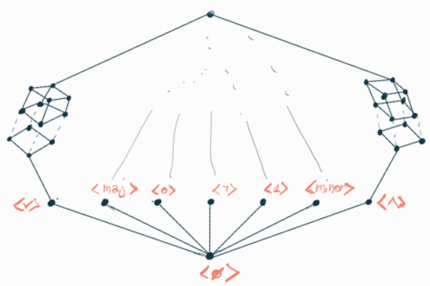
Thm: Let A be an n-element set. The set of all clones over A forms a lattice $\mathfrak{L}_n$ under inclusion.





$\mathfrak{L}_2$ a.k.a. Post Lattice

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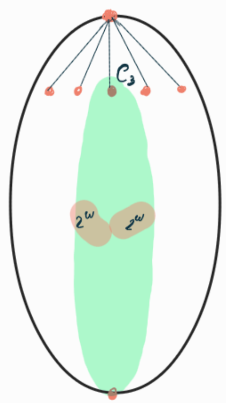
the clone of all self dual operations over $\{0,1,2\}$.



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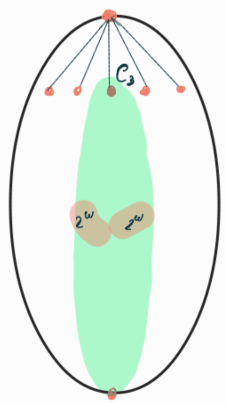

👁: C_3 is the only maximal clone over $\{0,1,2\}$, containing 2^{ω} many subclones whose lattice of subclones has been completely described.

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Formally, let f be an n-ary operation and $\pi : \{1, \dots, n\} \to \{1, \dots, r\}$.
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Def: A minor preserving map is a map $\xi : \mathcal{A} \to \mathcal{B}$ $(\mathcal{A} \leq_m \mathcal{B})$ such that
- ξ preserves arities;
- $\xi(f_\pi) = \xi(f)_\pi$, for any n-ary $f \in \mathcal{A}$ and $\pi : \{1, \dots, n\} \to \{1, \dots, r\}$.

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Def: We say that $\mathbb{B}$ is a pp-power of $\mathbb{A}$ if $\mathbb{B}$ is isomorphic to a structure $\mathbb{P}$ s.t.

- the domain of $\mathbb{P}$ is A^n, for some $n \geq 1$;
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Def: $\mathbb{A}$ pp-constructs $\mathbb{B}$ if $\mathbb{B}$ is hom. equivalent to a pp-power of $\mathbb{A}$.

CSP in a Nutshell

Def: $\mathbb{A}$, $\mathbb{B}$: τ-structures (τ: finite relational signature).

A homomorphism from $\mathbb{A}$ to $\mathbb{B}$ ($\mathbb{A} \to \mathbb{B}$) is a map $h: A \to B$ s.t., for every $R \in \tau$,

$$(a_1, \dots, a_n) \in R^{\mathbb{A}} \Longrightarrow (h(a_1), \dots, h(a_n)) \in R^{\mathbb{B}}$$

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Example:

- $\mathrm{CSP}(\mathbb{K}_3)$ is equivalent to the 3-colorability problem for graphs.

$\mathbb{K}_3 =$ 

- $\mathrm{CSP}(\underbrace{\{0, \dots, p-1\}}_{E_p};$ all affine subspaces $R_{abcd})$ ~ solving systems of lin. eq. over $\mathbb{Z}_p$

$$R_{abcd} := \{ (x, y, z) \in E_p^3 \mid ax + by + cz = d \}$$

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$$\forall x_1, \dots, x_n, y_1, \dots, y_m \left(f(\underbrace{x_1, \dots, x_n}) = g(\underbrace{y_1, \dots, y_m}) \right)$$

operation symbols

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For short:

$f(x_1, \dots, x_n) \approx g(y_1, \dots, y_m)$

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- A minor condition is a finite set of minor identities.

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Example:
- $m(x,x,y) \approx m(y,x,x) \approx m(y,y,y)$

  QUASI MAL'CEV
- $c(x_1, \dots, x_p) \approx c(x_2, \dots, x_p, x_1)$

  p-CYCLIC
- $m(x,x,y) \approx m(x,y,x) \approx m(y,x,x) \approx m(x,x,x)$

  QUASI MAJORITY
- $f(\underbrace{x_1, \dots, x_n}_{X}) \approx f(\underbrace{y_1, \dots, y_n}_{Y})$, whenever $X = Y$.

  TOT. SYMMETRIC

Algebra meets CSP

Def: • We say tha a set of operations F satisfies a minor condition Σ if there is a map ξ assigning to each operation symbol occurring in F an operation in F of the same arity s.t. if $(p \approx q) \in \Sigma$, then $\xi(p) = \xi(q)$. ($F \models \Sigma$)

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Thm [BOP '15]: Let $\mathbb{A}$ and $\mathbb{B}$ be finite relational structures. TFAE:

(1) There exist a minor-preserving map from Pol($\mathbb{A}$) to Pol($\mathbb{B}$) (Pol($\mathbb{A}$) $\leq_m$ Pol($\mathbb{B}$));

(2) $\mathbb{A}$ pp-constructs $\mathbb{B}$ ($\mathbb{A} \leq_{con} \mathbb{B}$);

(3) If Pol($\mathbb{A}$) $\models \Sigma$, then Pol($\mathbb{B}$) $\models \Sigma$.

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Thm [Siggers 2010]: Let $\mathcal{C}$ be a clone over a finite set. TFAE:

(1) $\mathcal{C}$ satisfies a non-trivial minor condition;

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Thm [Bulatov '17; Zhuk '17]: Let $\mathcal{A} = \mathrm{Pol}(\mathbb{A})$ be a clone over a finite set.

If $\mathcal{A} \models s(x,y,z,x,y,z) \approx s(y,x,x,z,z,y)$, then $\mathrm{CSP}(\mathbb{A})$ is in P.

Otherwise, $\mathrm{CSP}(\mathbb{A})$ is NP-complete

The pp-constructability poset

We write $\mathcal{C} \equiv_m \mathcal{D}$ iff $\mathcal{C} \leq_m \mathcal{D}$ and $\mathcal{D} \leq_m \mathcal{C}$.

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We denote by $\overline{\mathcal{C}}$ the $\equiv_m$-class of $\mathcal{C}$. Also, we write $\overline{\mathcal{C}} \leq_m \overline{\mathcal{D}}$ iff $\mathcal{C} \leq_m \mathcal{D}$.

$$\mathfrak{P}_{fin} := \left(\{ \overline{\mathcal{C}} \mid \mathcal{C} \text{ is a clone over some finite set} \} ; \leq_m \right)$$

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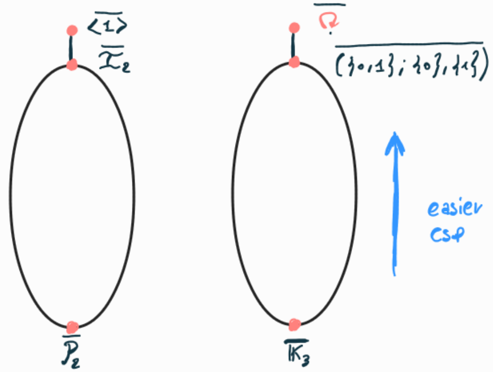
$$\mathfrak{P}_{fin} := (\{\overline{\mathcal{C}} \mid \mathcal{C} \text{ is a clone over some finite set}\}; \leq_m)$$

$$\mathfrak{P}_n := (\{\overline{\mathcal{C}} \mid \mathcal{C} \text{ is a clone over } \{0,1,\dots,n-1\}\}; \leq_m)$$



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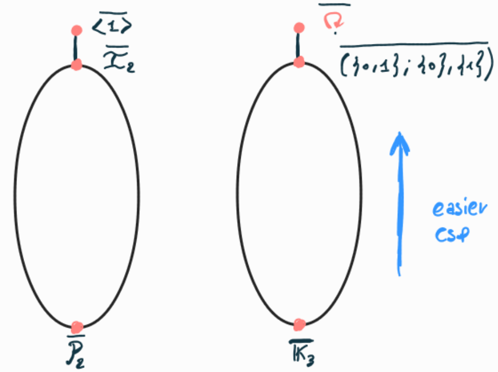
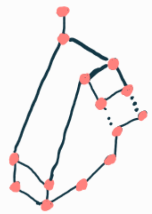


Some results: ▸ Complete description of $\mathfrak{P}_2$ [Bodirsky, V.]



The pp-constructability poset

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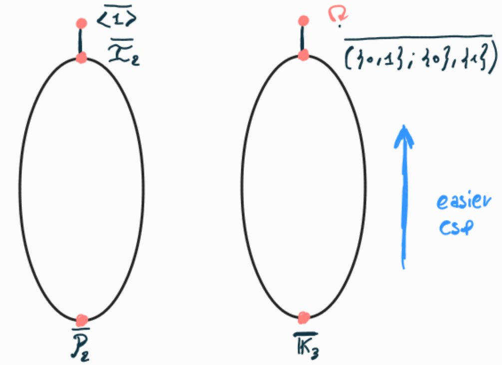

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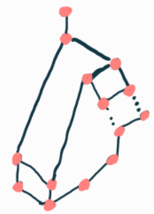

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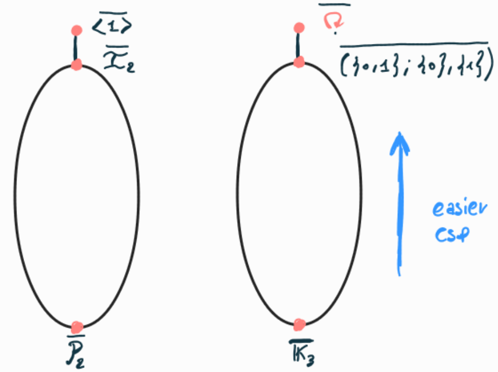

Some results:

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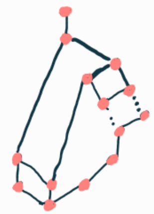

- Complete description of $\mathfrak{P}_{SD}$ (pol. clones of smooth digraphs) [Bodirsky, Starke, V.]
- Complete description of the lattice of clones of self-dual clones w.r.t. $\leq_m$ [Bodirsky, Zhuk, V.]

remarkably countably infinite!

Submaximal elements of $\mathcal{P}_3$

Want: Show that $\overline{\mathrm{Pol}(}$$\overline{)}$, $\overline{\mathrm{Pol}(}$$\overline{)}$, $\overline{\mathrm{Pol}(\{0,1\};\{(0,1),(1,0),(1,1)\},\{0\},\{1\})}$ are the only submax. el. of $\mathcal{P}_3$.

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What we really want: Show that if $\mathcal{A}$ is a clone over $\{0,1,2\}$ such that $\mathcal{A} \not\leq_m \mathcal{C}_2$, $\mathcal{A} \not\leq_m \mathcal{C}_3$, and $\mathcal{A} \not\leq_m \mathcal{B}_2$, then $\mathcal{I}_2 \leq_m \mathcal{A}$.

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Key observations:

- $\mathcal{I}_2 = \langle \wedge, x +_2 y +_2 z \rangle$
- $\forall n \geq 2$, $\wedge(x_1, \wedge(x_2, \wedge(x_3, \ldots \wedge(x_{n-1}, x_n)\ldots)))$ is a totally symm. op. of arity n.

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$\longrightarrow$ show that if $\mathcal{A}$ is a clone satisfying $(\circledast)$, then $\mathcal{A}$ has

- a tot. symm. op. of arity n, $\forall n \geq 2$;
- a operation that "simulates the odd sum modulo 2".

Submaximal elements of $\mathfrak{P}_3$

First step:

<u>Thm:</u> Let $\mathbb{A}$ be a finite structure. Then for every prime p:

$\mathbb{A}$ pp-contructs $\mathbb{C}_p$ $\iff$ $\mathrm{Pol}(\mathbb{A})$ does not satify $\underbrace{c(x_1,\dots,x_p) \approx c(x_2,\dots,x_p,x_1)}_{\Sigma_p}$

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proof (sketch): $(\Rightarrow)$ Suppose $\mathbb{A} \leq_{con} \mathbb{C}_p$, for some prime p.

If $\mathrm{Pol}(\mathbb{A}) \models \Sigma_p \overset{BOP}{\Longrightarrow} \mathrm{Pol}(\mathbb{C}_p) \models \Sigma_p$ i.e.

$$f(0,\dots,p-1) = c$$
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$\Rightarrow$ we get a pair (c,c) in $\mathbb{C}_p$ ↯

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$(\Longleftarrow)$ Suppose $\mathrm{Pol}(\mathbb{A}) \not\models \Sigma_p$. Want to build a pp-power of $\mathbb{A}$, which is hom. eq. to $\mathbb{C}_p$.

$\mathbb{P} = (P; R^{\mathbb{P}})$

- Domain of $\mathbb{P}$, $P = \{f \mid f: A^p \to A\}$
- Relation $R^{\mathbb{P}}$ is the following binary relation:
  it consists of all pairs (f,g) s.t. $f,g \in \mathrm{Pol}(\mathbb{A})$ and $f(x_1,\dots,x_p) = g(x_2,\dots,x_p,x_1)$

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- Partition $\mathrm{Pol}_p(\mathbb{A})$ into eq. classes s.t.

  $f_1 \sim f_2$ if f_1 can be obtained by a cyclic shift of the variables of f_2.

(to be continued) ↓

Submaximal elements of $\mathbb{P}_3$

proof: • Choose one representative from every $\sim$-class → denote this set of operations F_0.

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w applied to these tuples gives (f, g) where

$$f(x_1, \dots, x_p) = w\big(f_1(x_1, \dots, x_p), \dots, f_s(x_1, \dots, x_p)\big)$$
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$\Rightarrow$ f and g are from $Pol(A)$; moreover $g(x_1, \ldots, x_p) = f(x_2, \ldots, x_p, x_1)$

$\Rightarrow$ w preserves R^P $\Longrightarrow$ R^P is pp-definable over $\mathbb{A}$.

Submaximal elements of $\mathbb{P}_3$

proof: • Choose one representative from every $\sim$-class $\to$ denote this set of operations F_0.

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• Show that $\forall w \in Pol(\mathbb{A})$, w preserves $R^{\mathbb{P}}$. Consider $\underbrace{\begin{pmatrix} f_1 \\ g_1 \end{pmatrix}}_{\in R^{\mathbb{P}}}, \dots, \underbrace{\begin{pmatrix} f_s \\ g_s \end{pmatrix}}_{\in R^{\mathbb{P}}}$;

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• $\mathbb{P} \longleftrightarrow \mathbb{C}_p$:

$h : \mathbb{P} \to \mathbb{C}_p$
$F_i \mapsto i$, $\forall i$

$h' : \mathbb{C}_p \to \mathbb{P}$; choose $f_0 \in F_0$; all its cyclic shifts $f_i \in F_i$
$i \mapsto f_i$

$\square$

Submaximal elements of $\mathcal{P}_3$

Second step

Thm: Let $\mathbb{A}$ be a finite structure. $\mathbb{A}$ pp-constructs $\mathbb{B}_2$ $\Longleftrightarrow$ $\mathrm{Pol}(\mathbb{A})$ does not satisfy Mal'cev.

proof: (similar) ...

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Next steps: ▶ Step 1 + Step 2 $\Longrightarrow$ $\mathbb{A}$ has a $\underbrace{\text{2-cyclic, 3-cyclic}}$, and a Mal'cev operation.

$\xRightarrow{\text{BARTO, KOZIK}}$ p-cyclic, $\forall p$ prime.

Submaximal elements of $\mathbb{P}_3$

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- majority $\oplus$ Mal'cev $\Rightarrow$ $\mathbb{A}$ has a minority operation $\rightsquigarrow$ symmetric minority.
- symm. maj $\oplus$ minor. $\oplus$ 2-cyclic $\Rightarrow$ $\mathbb{A}$ has a totally symm. operation of arity n, $\forall n \geq 2$.

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$$S(x_1,\dots,x_n) := m\Big(S_{n-1}(x_1, M^c(x_1,x_2,x_3), x_4,\dots,x_n),\; S_{n-1}(x_2, M^c(x_1,x_2,x_3), x_4,\dots,x_n),\; S_{n-1}(x_1, M^c(x_1,x_2,x_3), x_4,\dots,x_n) \Big)$$

Submaximal elements of $\mathcal{P}_3$

The addition

Def: An oddition of arity n is an idempotent operation $m_n^c : A^n \to A$ s.t.

$$m_n^c(x_1, \dots, x_n) = \begin{cases} c \in A & \text{if there are at least three different values occurring in the tuple } (x_1, \dots, x_n) \\ \text{the only value occurring an odd number of times} & \text{otw.} \end{cases}$$

Submaximal elements of P_3

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- D. Zhuk $\Longrightarrow$ $m_{2n+1}^c := m_3^c($ ____ , ____ , ____ $)$

Submaximal elements of $\mathcal{P}_3$

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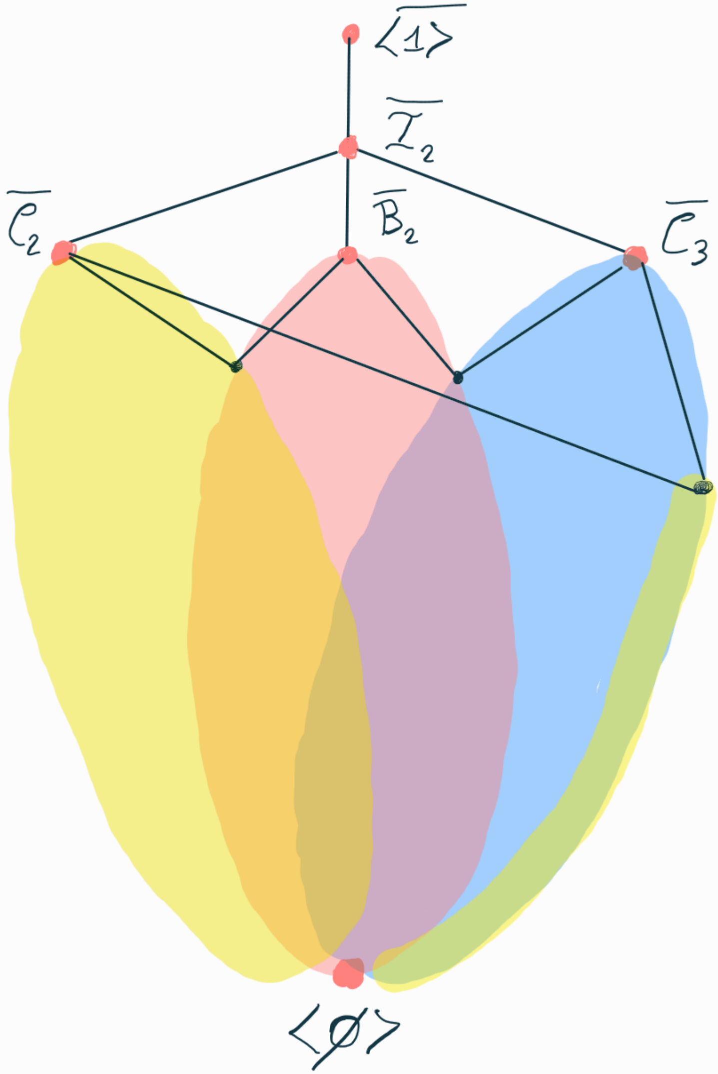
▶ D. Zhuk $\Rightarrow$ $m^c_{2n+1} := m^c_3(\underline{\qquad\qquad}, \underline{\qquad\qquad}, \underline{\qquad\qquad})$

Thm: There is a minor preserving map from $\mathbb{Z}_2$ to $\mathcal{A}$ (where $\mathcal{A}$ satisfies ⊛).

ξ : sum of ℓ monomials $\mapsto m_\ell(s_{n_1}(\ldots), \ldots, s_{n_\ell}(\ldots))$

$\mathcal{P}_3$: an overview



: fully described [B V Z]

: potentially 2^ω elements

: work in progress [FIORAVANTI, ROSSI, V.]


Thank You!