Clones over finite sets up to minor-equivalence

Albert Vucaj

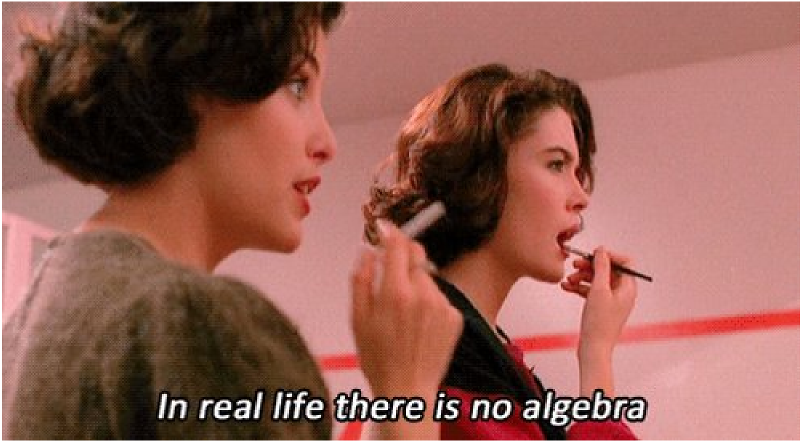
TU Wien

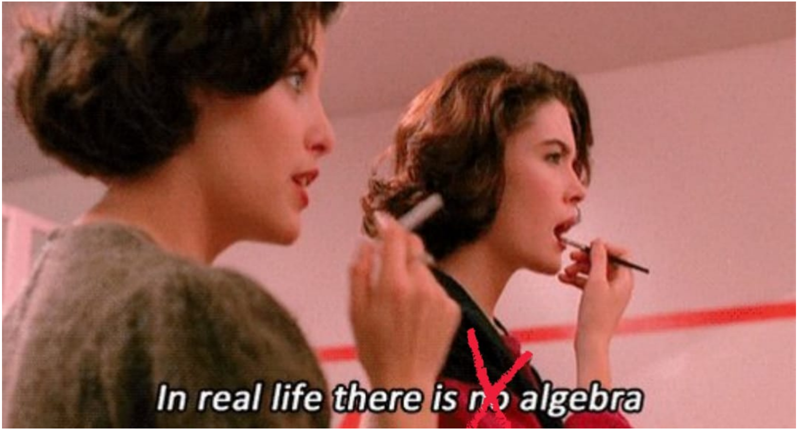
Algebra Week, Siena , 7 July 2023

European Research Council
Established by the European Commission

TCS4F
THEORETICAL COMPUTER SCIENTISTS FOR FUTURE

POCOCOP (ERC SyG No. 101071674)


In real life there is no algebra


In real life there is no algebra

TOPOLOGY IS IRRELEVANT
(IN A DICHOTOMY CONJECTURE FOR INFINITE DOMAIN CONSTRAINT SATISFACTION PROBLEMS)

LIBOR BARTO AND MICHAEL PINSKER

TOPOLOGY IS RELEVANT
(IN A DICHOTOMY CONJECTURE FOR INFINITE-DOMAIN CONSTRAINT SATISFACTION PROBLEMS)

MANUEL BODIRSKY, ANTOINE MOTTET, MIROSLAV OLŠÁK, JAKUB OPRŠAL, MICHAEL PINSKER, AND ROSS WILLARD

It is all about symmetries

It is all about symmetries

✉ **Message**: CSP($\mathbb{A}$) is hard $\iff$ $\mathbb{A}$ lacks symmetry.

It is all about symmetries

✉ **Message**: CSP($\mathbb{A}$) is hard $\iff$ $\mathbb{A}$ lacks symmetry.

- $\mathbb{A}, \mathbb{B}$: τ-structures (τ: finite relational signature).

Definition

A homomorphism from $\mathbb{A}$ to $\mathbb{B}$ is a map $h \colon A \to B$ s.t., for every $R \in \tau$,

$$(a_1, \ldots, a_n) \in R^{\mathbb{A}} \implies (h(a_1), \ldots, h(a_n)) \in R^{\mathbb{B}}.$$

In this case we write $\mathbb{A} \to \mathbb{B}$.

CSP($\mathbb{A}$) is the membership problem of the class

$$\{\mathbb{S} \mid \mathbb{S} \text{ is a } \tau\text{-structure and } \mathbb{S} \to \mathbb{A}\}.$$

It is all about symmetries

✉ **Message**: CSP($\mathbb{A}$) is hard $\iff$ $\mathbb{A}$ lacks symmetry.

- $\mathbb{A}, \mathbb{B}$: τ-structures (τ: finite relational signature).

Definition

A homomorphism from $\mathbb{A}$ to $\mathbb{B}$ is a map $h\colon A \to B$ s.t., for every $R \in \tau$,

$$(a_1, \ldots, a_n) \in R^{\mathbb{A}} \implies (h(a_1), \ldots, h(a_n)) \in R^{\mathbb{B}}.$$

In this case we write $\mathbb{A} \to \mathbb{B}$.

CSP($\mathbb{A}$) is the membership problem of the class

$$\{\mathbb{S} \mid \mathbb{S} \text{ is a } \tau\text{-structure and } \mathbb{S} \to \mathbb{A}\}.$$

Example

CSP($\mathbb{K}_3$) *is equivalent to the 3-colorability problem.*

It is all about symmetries

- $\mathbb{A}$: τ-structure;
- $\phi(x_1, \ldots, x_n)$: a τ-formula with n free-variables $x_1, \ldots, x_n$.

Definition

We call $R = \{(a_1, \ldots, a_n) \mid \mathbb{A} \models \phi(a_1, \ldots, a_n)\}$ *the relation defined by* ϕ.

It is all about symmetries

- $\mathbb{A}$: τ-structure;
- $\phi(x_1, \ldots, x_n)$: a τ-formula with n free-variables $x_1, \ldots, x_n$.

Definition

We call $R = \{(a_1, \ldots, a_n) \mid \mathbb{A} \models \phi(a_1, \ldots, a_n)\}$ the relation defined by ϕ.

If ϕ is primitive positive, then R is said to be pp-definable in $\mathbb{A}$.

It is all about symmetries

- $\mathbb{A}$: τ-structure;
- $\phi(x_1, \ldots, x_n)$: a τ-formula with n free-variables $x_1, \ldots, x_n$.

Definition

We call $R = \{(a_1, \ldots, a_n) \mid \mathbb{A} \models \phi(a_1, \ldots, a_n)\}$ the relation defined by ϕ.

If ϕ is primitive positive, then R is said to be pp-definable in $\mathbb{A}$.

Definition

$\mathbb{B}$ is a pp-power of $\mathbb{A}$ if $\mathbb{B}$ is isomorphic to a structure $\mathbb{P}$ such that

- *the domain of $\mathbb{P}$ is A^n, $n \geq 1$;*
- *all the relations of $\mathbb{P}$ are pp-definable from $\mathbb{A}$.*

It is all about symmetries

- $\mathbb{A}$: τ-structure;
- $\phi(x_1, \ldots, x_n)$: a τ-formula with n free-variables $x_1, \ldots, x_n$.

Definition

We call $R = \{(a_1, \ldots, a_n) \mid \mathbb{A} \models \phi(a_1, \ldots, a_n)\}$ the relation defined by ϕ.

If ϕ is primitive positive, then R is said to be pp-definable in $\mathbb{A}$.

Definition

$\mathbb{B}$ is a pp-power of $\mathbb{A}$ if $\mathbb{B}$ is isomorphic to a structure $\mathbb{P}$ such that

- *the domain of $\mathbb{P}$ is A^n, $n \geq 1$;*
- *all the relations of $\mathbb{P}$ are pp-definable from $\mathbb{A}$.*

Definition

$\mathbb{A}$ pp-constructs $\mathbb{B}$ if $\mathbb{B}$ is homomorphically equivalent to a pp-power of $\mathbb{A}$.

It is all about symmetries

Theorem (Barto, Opršal, Pinsker 2015)

If $\mathbb{A}$ *pp-constructs* $\mathbb{B}$*, then* $\mathsf{CSP}(\mathbb{B}) \leq_{\log} \mathsf{CSP}(\mathbb{A})$.

It is all about symmetries

Theorem (Barto, Opršal, Pinsker 2015)

If $\mathbb{A}$ *pp-constructs* $\mathbb{B}$, *then* $\mathrm{CSP}(\mathbb{B}) \leq_{\log} \mathrm{CSP}(\mathbb{A})$.

Aut($\mathbb{A}$) is NOT the right notion of symmetry!
For every finite structure $\mathbb{A}$, there exists a finite structure $\mathbb{B}$ s.t.:

- $\mathbb{A}$ and $\mathbb{B}$ pp-construct each other (same complexity)
- **Aut**($\mathbb{B}$) $= \{\mathrm{id}_B\}$.

It is all about symmetries

Theorem (Barto, Opršal, Pinsker 2015)

If $\mathbb{A}$ *pp-constructs* $\mathbb{B}$*, then* $\mathrm{CSP}(\mathbb{B}) \leq_{\log} \mathrm{CSP}(\mathbb{A})$.

- **Aut**$(\mathbb{A})$ is NOT the right notion of symmetry!
  For every finite structure $\mathbb{A}$, there exists a finite structure $\mathbb{B}$ s.t.:
  - $\mathbb{A}$ and $\mathbb{B}$ pp-construct each other (same complexity)
  - **Aut**$(\mathbb{B}) = \{\mathrm{id}_B\}$.

- Since $\mathrm{CSP}(\mathbb{K}_3)$ is NP-complete: if $\mathbb{A}$ pp-constructs $\mathbb{K}_3$, then $\mathrm{CSP}(\mathbb{A})$ in NP-complete.

It is all about symmetries

Theorem (Barto, Opršal, Pinsker 2015)

If $\mathbb{A}$ *pp-constructs* $\mathbb{B}$*, then* $\mathrm{CSP}(\mathbb{B}) \leq_{\log} \mathrm{CSP}(\mathbb{A})$*.*

- **Aut**$(\mathbb{A})$ is NOT the right notion of symmetry!
  For every finite structure $\mathbb{A}$, there exists a finite structure $\mathbb{B}$ s.t.:
  - $\mathbb{A}$ and $\mathbb{B}$ pp-construct each other (same complexity)
  - **Aut**$(\mathbb{B}) = \{\mathrm{id}_B\}$.

- Since $\mathrm{CSP}(\mathbb{K}_3)$ is NP-complete: if $\mathbb{A}$ pp-constructs $\mathbb{K}_3$, then $\mathrm{CSP}(\mathbb{A})$ in NP-complete.

Reason: $\mathbb{K}_3$ has few symmetries.

It is all about symmetries

Definition

An operation $f : A^n \to A$ preserves a k-ary relation R on A if

$$\begin{matrix} a_{1,1} & a_{1,2} & & a_{1,n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{k,1} & a_{k,2} & & a_{k,n} \\ \in & \in & & \in \\ R & R & & R \end{matrix}$$

In this case, we also say that R is invariant under f.

It is all about symmetries

Definition

An operation $f\colon A^n \to A$ preserves a k-ary relation R on A if

$$\begin{matrix} f(a_{1,1} & a_{1,2} & & a_{1,n}) \\ \vdots & \vdots & \cdots & \vdots \\ f(a_{k,1} & a_{k,2} & & a_{k,n}) \\ \in & \in & & \in \\ R & R & & R \end{matrix}$$

In this case, we also say that R is invariant under f.

It is all about symmetries

Definition

An operation $f: A^n \to A$ preserves a k-ary relation R on A if

$$
\begin{matrix}
f(a_{1,1} & a_{1,2} & & a_{1,n}) \\
\vdots & \vdots & \cdots & \vdots \\
f(a_{k,1} & a_{k,2} & & a_{k,n})
\end{matrix}
\quad
\begin{matrix} \in \\ R \end{matrix}
$$

In this case, we also say that R is invariant under f.

It is all about symmetries

Definition

An operation $f\colon A^n \to A$ preserves a k-ary relation R on A if

$$\begin{array}{cccc} f(a_{1,1} & a_{1,2} & & a_{1,n}) \\ \vdots & \vdots & \cdots & \vdots \\ f(a_{k,1} & a_{k,2} & & a_{k,n}) \end{array} \in R$$

In this case, we also say that R is invariant under f.

Definition

- *f is a polymorphism of $\mathbb{A} = (A; \Gamma)$ if f preserves R, for every $R \in \Gamma$.*

It is all about symmetries

Definition

An operation $f : A^n \to A$ preserves a k-ary relation R on A if

$$\begin{matrix} f(a_{1,1} & a_{1,2} & & a_{1,n}) \\ \vdots & \vdots & \cdots & \vdots \\ f(a_{k,1} & a_{k,2} & & a_{k,n}) \end{matrix} \underset{R}{\in}$$

In this case, we also say that R is invariant under f.

Definition

- *f is a polymorphism of $\mathbb{A} = (A; \Gamma)$ if f preserves R, for every $R \in \Gamma$.*
- $\mathrm{Pol}(\mathbb{A}) = \{f \mid f$ *is a polymorphism of* $\mathbb{A}\}$ *(the polym. clone of $\mathbb{A}$).*

It is all about symmetries

Definition

An operation $f: A^n \to A$ preserves a k-ary relation R on A if

$$\begin{matrix} f(a_{1,1} & a_{1,2} & & a_{1,n}) \\ \vdots & \vdots & \cdots & \vdots \\ f(a_{k,1} & a_{k,2} & & a_{k,n}) \end{matrix} \in R$$

In this case, we also say that R is invariant under f.

Definition

- *f is a polymorphism of $\mathbb{A} = (A; \Gamma)$ if f preserves R, for every $R \in \Gamma$.*
- $\mathrm{Pol}(\mathbb{A}) = \{f \mid f$ *is a polymorphism of* $\mathbb{A}\}$ *(the polym. clone of $\mathbb{A}$).*
- $\mathrm{Inv}(F) = \{R \mid R$ *is invariant under every operation in* $F\}$.

Clones

Clones

Definition

A clone $\mathcal{C}$ is a set of operations over a finite set A such that

- *$\mathcal{C}$ contains all the projections,*
- *$\mathcal{C}$ is closed under composition.*

Clones

Definition

A clone $\mathcal{C}$ *is a set of operations over a finite set* A *such that*

- $\mathcal{C}$ *contains all the projections,*
- $\mathcal{C}$ *is closed under composition.*

If F *is a set of operations, we denote by* $\langle F \rangle$ *the clone generated by* F.

Clones

Definition

A clone $\mathcal{C}$ is a set of operations over a finite set A such that

- *$\mathcal{C}$ contains all the projections,*
- *$\mathcal{C}$ is closed under composition.*

If F is a set of operations, we denote by $\langle F \rangle$ the clone generated by F.

Example

Consider the universe $\{0, 1\}$:

Clones

Definition

A clone $\mathcal{C}$ is a set of operations over a finite set A such that

- *$\mathcal{C}$ contains all the projections,*
- *$\mathcal{C}$ is closed under composition.*

If F is a set of operations, we denote by $\langle F \rangle$ the clone generated by F.

Example

Consider the universe $\{0, 1\}$:

- $\mathcal{P}_2 := \langle \emptyset \rangle$ *(the clone of all projections on $\{0, 1\}$);*

Clones

Definition

A clone $\mathcal{C}$ is a set of operations over a finite set A such that

- *$\mathcal{C}$ contains all the projections,*
- *$\mathcal{C}$ is closed under composition.*

If F is a set of operations, we denote by $\langle F \rangle$ the clone generated by F.

Example

Consider the universe $\{0, 1\}$:

- $\mathcal{P}_2 := \langle \emptyset \rangle$ *(the clone of all projections on $\{0, 1\}$);*
- $\mathcal{I}_2 := \langle \wedge, m \rangle$ *(the clone of all idempotent operations on $\{0, 1\}$).*

Clones

Definition

A clone $\mathcal{C}$ is a set of operations over a finite set A such that

- *$\mathcal{C}$ contains all the projections,*
- *$\mathcal{C}$ is closed under composition.*

If F is a set of operations, we denote by $\langle F \rangle$ the clone generated by F.

Example

Consider the universe $\{0, 1\}$:

- $\mathcal{P}_2 := \langle \emptyset \rangle$ *(the clone of all projections on $\{0, 1\}$);*
- $\mathcal{I}_2 := \langle \wedge, m \rangle$ *(the clone of all idempotent operations on $\{0, 1\}$).*

Theorem (Geiger '68; Bodnarčuk, Kalužnin, Kotov, Romov '69)

If F is a set of operations on a finite domain, then $\mathrm{Pol}(\mathrm{Inv}(F)) = \langle F \rangle$.

A Galois connection for clones

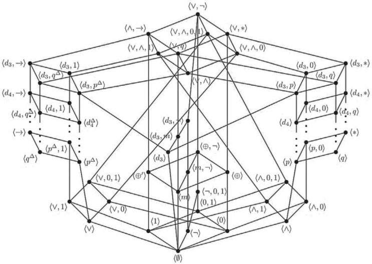
Corollary

All clones over a finite n-element set form a lattice $\mathfrak{L}_n$ *under inclusion.*

A Galois connection for clones

Corollary

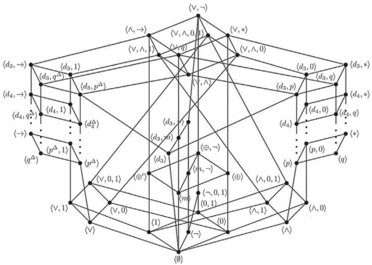
All clones over a finite n-element set form a lattice $\mathfrak{L}_n$ under inclusion.



A Galois connection for clones

Corollary

All clones over a finite n-element set form a lattice $\mathfrak{L}_n$ under inclusion.



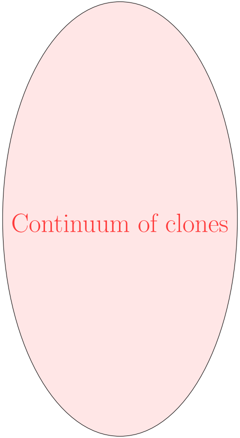
Theorem

- $\mathbb{A}$, $\mathbb{B}$: *relational structures on the same finite universe* A,
- $\mathcal{A} = \mathrm{Pol}(\mathbb{A})$ *and* $\mathcal{B} = \mathrm{Pol}(\mathbb{B})$.

$\mathbb{A}$ *pp-defines* $\mathbb{B} \iff \mathcal{A} \subseteq \mathcal{B}$.

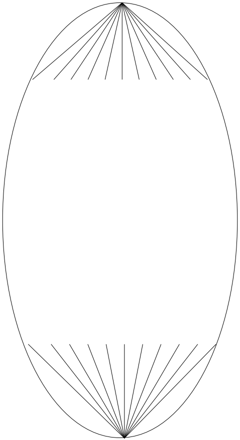
Clones over $\{0, 1, 2\}$

Clones over $\{0, 1, 2\}$



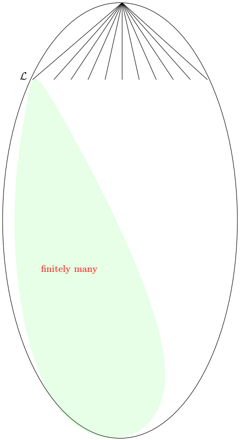
☹ There exists a continuum of clones over $\{0, 1, 2\}$ (Yanov, Muchnik '59).

Clones over $\{0, 1, 2\}$



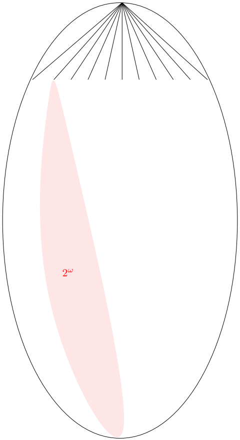
☺ Description of all maximal and minimal clones.
(Jablonskij '54; Csákány '83)

Clones over $\{0, 1, 2\}$



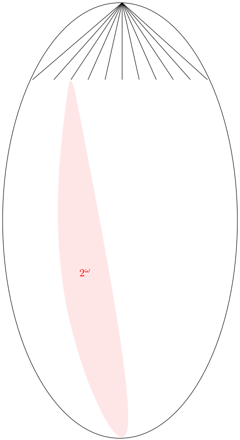
☹ All maximal clones – except the clone of all linear functions – contain a continuum of subclones (Demetrovics, Hannak '83; Marchenkov '83).

Clones over $\{0, 1, 2\}$



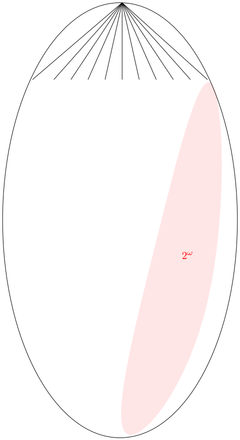
☹ All maximal clones – except the clone of all linear functions – contain a continuum of subclones (Demetrovics, Hannak '83; Marchenkov '83).

Clones over $\{0, 1, 2\}$



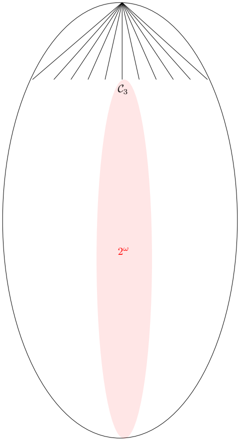
☹ All maximal clones – except the clone of all linear functions – contain a continuum of subclones (Demetrovics, Hannak '83; Marchenkov '83).

Clones over $\{0, 1, 2\}$



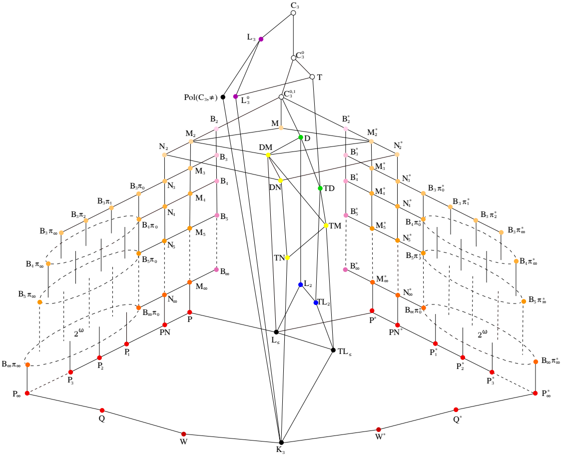
☹ All maximal clones – except the clone of all linear functions – contain a continuum of subclones (Demetrovics, Hannak '83; Marchenkov '83).

Clones over $\{0, 1, 2\}$



☺ D. Zhuk: "Continuum is not a problem" (2015).

Clones over $\{0, 1, 2\}$



☺ D. Zhuk: "Continuum is not a problem" (2015).

Coffee break!

A new order

What we want: $\mathbb{A}$ pp-constructs $\mathbb{B} \iff \mathrm{Pol}(\mathbb{A})$?? $\mathrm{Pol}(\mathbb{B})$.

A new order

What we want: $\mathbb{A}$ pp-constructs $\mathbb{B}$ $\Longleftrightarrow$ $\mathrm{Pol}(\mathbb{A})$?? $\mathrm{Pol}(\mathbb{B})$.

Definition

- τ: *set of function symbols;*
  A minor identity (height 1 identity) is an identity of the form

$$f(x_1,\dots,x_n) \approx g(y_1,\dots,y_m)$$

  where $f, g \in \tau$ and $x_1,\dots,x_n,y_1,\dots,y_m$ are not necessarily distinct.
- *Minor condition: Finite set of minor identities.*

A new order

What we want: $\mathbb{A}$ pp-constructs $\mathbb{B} \iff \mathrm{Pol}(\mathbb{A})$?? $\mathrm{Pol}(\mathbb{B})$.

Definition

- τ: *set of function symbols;*
  A minor identity (height 1 identity) is an identity of the form

$$f(x_1, \ldots, x_n) \approx g(y_1, \ldots, y_m)$$

  where $f, g \in \tau$ and $x_1, \ldots, x_n, y_1, \ldots, y_m$ are not necessarily distinct.
- *Minor condition: Finite set of minor identities.*

Example

$f(x, y) \approx f(y, x)$ ✔

$f(f(x, y), z) \approx f(x, f(y, z))$ ✖

$m(x, x, y) \approx m(y, x, x) \approx y$ ✖ *(Mal'cev)*

A new order

What we want: $\mathbb{A}$ pp-constructs $\mathbb{B} \iff \mathrm{Pol}(\mathbb{A})$?? $\mathrm{Pol}(\mathbb{B})$.

Definition

- *τ: set of function symbols;*
  A minor identity (height 1 identity) is an identity of the form

  $$f(x_1, \ldots, x_n) \approx g(y_1, \ldots, y_m)$$

  where $f, g \in \tau$ and $x_1, \ldots, x_n, y_1, \ldots, y_m$ are not necessarily distinct.
- *Minor condition: Finite set of minor identities.*

Example

$$f(x,y) \approx f(y,x) \ ✔$$

$$f(f(x,y),z) \approx f(x, f(y,z)) \ ✖$$

$$m(x,x,y) \approx m(y,x,x) \approx m(y,y,y) \ ✔$$

(quasi Mal'cev)

A new order

Definition

We say that F satisfies Σ ($F \models \Sigma$) if there is a map ξ assigning to each function symbol occurring in Σ an operation in F of the same arity, such that if $p \approx q$ is in Σ, then $\xi(p) = \xi(q)$.

A new order

Definition

We say that F satisfies Σ *(*$F \models \Sigma$*) if there is a map* ξ *assigning to each function symbol occurring in* Σ *an operation in F of the same arity, such that if* $p \approx q$ *is in* Σ*, then* $\xi(p) = \xi(q)$*.*

A minor condition is trivial if it is satisfied by $\mathcal{P}_2$.

A new order

Definition

We say that F satisfies Σ ($F \models \Sigma$) if there is a map ξ assigning to each function symbol occurring in Σ an operation in F of the same arity, such that if $p \approx q$ is in Σ, then $\xi(p) = \xi(q)$.

A minor condition is trivial if it is satisfied by $\mathcal{P}_2$.

: $\mathrm{Pol}(\mathbb{K}_3)$ does not satisfy any non-trivial minor condition.

A new order

Definition

We say that F satisfies Σ ($F \models \Sigma$) if there is a map ξ assigning to each function symbol occurring in Σ an operation in F of the same arity, such that if $p \approx q$ is in Σ, then $\xi(p) = \xi(q)$.

A minor condition is trivial if it is satisfied by $\mathcal{P}_2$.

: $\mathrm{Pol}(\mathbb{K}_3)$ does not satisfy any non-trivial minor condition.
Equivalently: $\mathrm{Pol}(\mathbb{K}_3)$ does not satisfy

$$s(x, y, z, x, y, z) \approx s(y, x, x, z, z, y).$$

Minors and Reflections

Let f be any n-ary operation and $\sigma\colon \{1,\ldots,n\} \to \{1,\ldots,r\}$.

Minors and Reflections

Let f be any n-ary operation and $\sigma\colon \{1,\dots,n\} \to \{1,\dots,r\}$.
We write f_σ to denote $f_\sigma(x_1,\dots,x_r) := f(x_{\sigma(1)},\dots,x_{\sigma(n)})$.
Any operation of the form f_σ is called a minor of f.

Minors and Reflections

Let f be any n-ary operation and $\sigma\colon \{1,\ldots,n\} \to \{1,\ldots,r\}$.
We write f_σ to denote $f_\sigma(x_1,\ldots,x_r) := f(x_{\sigma(1)},\ldots,x_{\sigma(n)})$.
Any operation of the form f_σ is called a minor of f.

Definition

A minor-preserving map is a map $\xi\colon \mathcal{A} \to \mathcal{B}$ such that

- *ξ preserves arities;*
- *$\xi(f_\sigma) = \xi(f)_\sigma$ for any n-ary operation $f \in \mathcal{A}$ and $\sigma\colon E_n \to E_r$.*

It is a weakening of the notion of clone homomorphism.

Minors and Reflections

Let f be any n-ary operation and $\sigma\colon \{1,\ldots,n\} \to \{1,\ldots,r\}$.
We write f_σ to denote $f_\sigma(x_1,\ldots,x_r) := f(x_{\sigma(1)},\ldots,x_{\sigma(n)})$.
Any operation of the form f_σ is called a minor of f.

Definition

A minor-preserving map is a map $\xi\colon \mathcal{A} \to \mathcal{B}$ such that

- *ξ preserves arities;*
- *$\xi(f_\sigma) = \xi(f)_\sigma$ for any n-ary operation $f \in \mathcal{A}$ and $\sigma\colon E_n \to E_r$.*

It is a weakening of the notion of clone homomorphism.

Theorem (Birkhoff, 1935)

Let $\mathcal{A}$, $\mathcal{B}$ be clones over finite sets. The following are equivalent:

1. *There exists a clone homomorphism from $\mathcal{A}$ to $\mathcal{B}$;*
2. *$\mathcal{B} \in \boldsymbol{EHSP}_{\text{fin}}(\mathcal{A})$.*

Minors and Reflections

Let f be any n-ary operation and $\sigma\colon \{1,\dots,n\} \to \{1,\dots,r\}$.
We write f_σ to denote $f_\sigma(x_1,\dots,x_r) := f(x_{\sigma(1)},\dots,x_{\sigma(n)})$.
Any operation of the form f_σ is called a minor of f.

Definition

A minor-preserving map is a map $\xi\colon \mathcal{A} \to \mathcal{B}$ such that

- *ξ preserves arities;*
- *$\xi(f_\sigma) = \xi(f)_\sigma$ for any n-ary operation $f \in \mathcal{A}$ and $\sigma\colon E_n \to E_r$.*

It is a weakening of the notion of clone homomorphism.

Theorem (Barto, Opršal, Pinsker, 2015)

Let $\mathcal{A}$, $\mathcal{B}$ be clones over finite sets. The following are equivalent:

1. *There exists a minor-preserving map from $\mathcal{A}$ to $\mathcal{B}$ ($\mathcal{A} \leq_{\mathrm{m}} \mathcal{B}$);*
2. *$\mathcal{B} \in \boldsymbol{ERP}_{\mathrm{fin}}(\mathcal{A})$.*

Minors and Reflections

Let f be any n-ary operation and $\sigma\colon \{1,\dots,n\} \to \{1,\dots,r\}$.
We write f_σ to denote $f_\sigma(x_1,\dots,x_r) := f(x_{\sigma(1)},\dots,x_{\sigma(n)})$.
Any operation of the form f_σ is called a minor of f.

Definition

A minor-preserving map is a map $\xi\colon \mathcal{A} \to \mathcal{B}$ such that

- *ξ preserves arities;*
- *$\xi(f_\sigma) = \xi(f)_\sigma$ for any n-ary operation $f \in \mathcal{A}$ and $\sigma\colon E_n \to E_r$.*

It is a weakening of the notion of clone homomorphism.

Theorem (Barto, Opršal, Pinsker, 2015)

Let $\mathcal{A}$, $\mathcal{B}$ be clones over finite sets. The following are equivalent:

1. *There exists a minor-preserving map from $\mathcal{A}$ to $\mathcal{B}$ ($\mathcal{A} \leq_{\mathrm{m}} \mathcal{B}$);*
2. *$\mathcal{B} \in \boldsymbol{ERP}_{\mathrm{fin}}(\mathcal{A})$.*

NO GALOIS CONNECTION

Motivation: CSP

Theorem (Barto, Opršal, Pinsker, 2015)

Let $\mathbb{A}$, $\mathbb{B}$ *be finite relational structures;* $\mathcal{A} = \mathrm{Pol}(\mathbb{A})$, $\mathcal{B} = \mathrm{Pol}(\mathbb{B})$. *TFAE:*

1. *There exists a minor-preserving map from* $\mathcal{A}$ *to* $\mathcal{B}$ *(*$\mathcal{A} \leq_{\mathrm{m}} \mathcal{B}$*);*
2. $\mathbb{A}$ *pp-constructs* $\mathbb{B}$ *(*$\mathbb{A} \leq_{\mathrm{Con}} \mathbb{B}$*);*
3. *if* $\mathcal{A}$ *satisfies a minor condition* Σ, *then* $\mathcal{B} \models \Sigma$.

Motivation: CSP

Theorem (Barto, Opršal, Pinsker, 2015)

Let $\mathbb{A}$, $\mathbb{B}$ *be finite relational structures;* $\mathcal{A} = \mathrm{Pol}(\mathbb{A})$, $\mathcal{B} = \mathrm{Pol}(\mathbb{B})$. *TFAE:*

1. *There exists a minor-preserving map from* $\mathcal{A}$ *to* $\mathcal{B}$ ($\mathcal{A} \leq_{\mathrm{m}} \mathcal{B}$);
2. $\mathbb{A}$ *pp-constructs* $\mathbb{B}$ ($\mathbb{A} \leq_{\mathrm{Con}} \mathbb{B}$);
3. *if* $\mathcal{A}$ *satisfies a minor condition* Σ, *then* $\mathcal{B} \models \Sigma$.

Great achievement: CSP Dichotomy Theorem!

- positive solution to the Feder-Vardi conjecture, open since 1998;
- new algebraic theories for finite algebras (Absorption, Bulatov-edges, strong subalgebras,...)

Theorem (Bulatov 2017; Zhuk 2017)

If there is no minor-preserving map from $\mathcal{A}$ *to* $\mathcal{P}_2$, *then* $\mathrm{CSP}(\mathbb{A})$ *is in P. Otherwise,* $\mathrm{CSP}(\mathbb{A})$ *is NP-complete*

Motivation: CSP

Theorem (Barto, Opršal, Pinsker, 2015)

Let $\mathbb{A}$, $\mathbb{B}$ be finite relational structures; $\mathcal{A} = \mathrm{Pol}(\mathbb{A})$, $\mathcal{B} = \mathrm{Pol}(\mathbb{B})$. TFAE:

1. *There exists a minor-preserving map from $\mathcal{A}$ to $\mathcal{B}$ ($\mathcal{A} \leq_{\mathrm{m}} \mathcal{B}$);*
2. *$\mathbb{A}$ pp-constructs $\mathbb{B}$ ($\mathbb{A} \leq_{\mathrm{Con}} \mathbb{B}$);*
3. *if $\mathcal{A}$ satisfies a minor condition Σ, then $\mathcal{B} \models \Sigma$.*

Great achievement: CSP Dichotomy Theorem!

- positive solution to the Feder-Vardi conjecture, open since 1998;
- new algebraic theories for finite algebras (Absorption, Bulatov-edges, strong subalgebras,...)

Theorem (Bulatov 2017; Zhuk 2017)

If $\mathbb{A}$ does not pp-construct $\mathbb{K}_3 = (\{0,1,2\}; \neq)$, then $\mathrm{CSP}(\mathbb{A})$ is in P. Otherwise, $\mathrm{CSP}(\mathbb{A})$ is NP-complete

Algebra meets CSP

Theorem (Barto, Opršal, Pinsker, 2015)

Let $\mathbb{A}$, $\mathbb{B}$ be finite relational structures; $\mathcal{A} = \mathrm{Pol}(\mathbb{A})$, $\mathcal{B} = \mathrm{Pol}(\mathbb{B})$. TFAE:

1. *There exists a minor-preserving map from $\mathcal{A}$ to $\mathcal{B}$ ($\mathcal{A} \leq_{\mathrm{m}} \mathcal{B}$);*
2. *$\mathbb{A}$ pp-constructs $\mathbb{B}$ ($\mathbb{A} \leq_{\mathrm{Con}} \mathbb{B}$);*
3. *if $\mathcal{A}$ satisfies a minor condition Σ, then $\mathcal{B} \models \Sigma$.*

Great achievement: CSP Dichotomy Theorem!

- positive solution to the Feder-Vardi conjecture, open since 1998;
- new algebraic theories for finite algebras (Absorption, Bulatov-edges, strong subalgebras,...)

Theorem (Bulatov 2017; Zhuk 2017)

If $\mathcal{A}$ satisfies a non-trivial minor condition, then $\mathrm{CSP}(\mathbb{A})$ is in P. Otherwise, $\mathrm{CSP}(\mathbb{A})$ is NP-complete

The pp-constructability poset

- $\leq_{\mathrm{m}}$ is a quasi order.

The pp-constructability poset

- $\leq_{\mathrm{m}}$ is a quasi order.
- We write $\mathcal{C} \equiv_{\mathrm{m}} \mathcal{D}$ iff $\mathcal{C} \leq_{\mathrm{m}} \mathcal{D}$ and $\mathcal{D} \leq_{\mathrm{m}} \mathcal{C}$. (minor-equivalent)

The pp-constructability poset

- $\leq_{\mathrm{m}}$ is a quasi order.
- We write $\mathcal{C} \equiv_{\mathrm{m}} \mathcal{D}$ iff $\mathcal{C} \leq_{\mathrm{m}} \mathcal{D}$ and $\mathcal{D} \leq_{\mathrm{m}} \mathcal{C}$. (minor-equivalent)
- $\overline{\mathcal{C}}$ is the $\equiv_{\mathrm{m}}$-class of $\mathcal{C}$.

The pp-constructability poset

- $\leq_{\mathrm{m}}$ is a quasi order.
- We write $\mathcal{C} \equiv_{\mathrm{m}} \mathcal{D}$ iff $\mathcal{C} \leq_{\mathrm{m}} \mathcal{D}$ and $\mathcal{D} \leq_{\mathrm{m}} \mathcal{C}$. (minor-equivalent)
- $\overline{\mathcal{C}}$ is the $\equiv_{\mathrm{m}}$-class of $\mathcal{C}$.

Definition

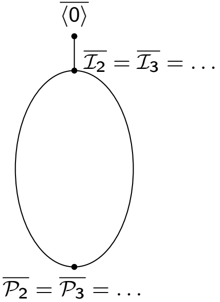
$$\mathfrak{P}_{\mathrm{fin}} := \left(\{\overline{\mathcal{C}} \mid \mathcal{C} \textit{ is a clone over some finite set}\}; \leq_{\mathrm{m}}\right)$$
$$\mathfrak{P}_n := \left(\{\overline{\mathcal{C}} \mid \mathcal{C} \textit{ is a clone over } \{0, \ldots, n-1\}\}; \leq_{\mathrm{m}}\right)$$

The pp-constructability poset

- $\leq_{\mathrm{m}}$ is a quasi order.
- We write $\mathcal{C} \equiv_{\mathrm{m}} \mathcal{D}$ iff $\mathcal{C} \leq_{\mathrm{m}} \mathcal{D}$ and $\mathcal{D} \leq_{\mathrm{m}} \mathcal{C}$. (minor-equivalent)
- $\overline{\mathcal{C}}$ is the $\equiv_{\mathrm{m}}$-class of $\mathcal{C}$.

Definition

$$\mathfrak{P}_{\mathrm{fin}} := \left(\{\overline{\mathcal{C}} \mid \mathcal{C} \textit{ is a clone over some finite set}\}; \leq_{\mathrm{m}}\right)$$
$$\mathfrak{P}_n := \left(\{\overline{\mathcal{C}} \mid \mathcal{C} \textit{ is a clone over } \{0,\dots,n-1\}\}; \leq_{\mathrm{m}}\right)$$

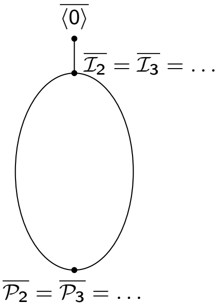


The pp-constructability poset

- $\leq_{\mathrm{m}}$ is a quasi order. ($\leq_{\mathrm{Con}}$ is a quasi order)
- We write $\mathbb{C} \equiv_{\mathrm{Con}} \mathbb{D}$ iff $\mathbb{C} \leq_{\mathrm{Con}} \mathbb{D}$ and $\mathbb{D} \leq_{\mathrm{Con}} \mathbb{C}$. (pp-equivalent)
- $\overline{\mathcal{C}}$ is the $\equiv_{\mathrm{m}}$-class of $\mathcal{C}$ ($\overline{\mathbb{C}}$ is the $\equiv_{\mathrm{Con}}$-class of $\mathbb{C}$).

Definition

$$\mathfrak{P}_{\mathrm{fin}} := \left(\{\overline{\mathcal{C}} \mid \mathcal{C} \textit{ is a clone over some finite set}\}; \leq_{\mathrm{m}}\right)$$
$$\mathfrak{P}_n := \left(\{\overline{\mathcal{C}} \mid \mathcal{C} \textit{ is a clone over } \{0, \dots, n-1\}\}; \leq_{\mathrm{m}}\right)$$

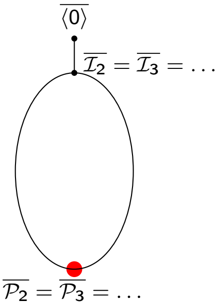


The pp-constructability poset

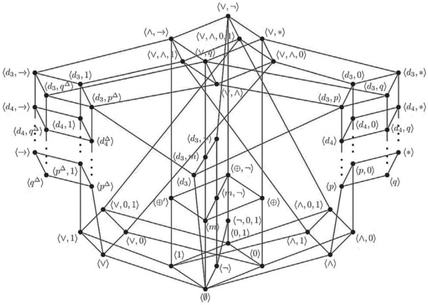
- $\leq_{\mathrm{m}}$ is a quasi order. ($\leq_{\mathrm{Con}}$ is a quasi order)
- We write $\mathbb{C} \equiv_{\mathrm{Con}} \mathbb{D}$ iff $\mathbb{C} \leq_{\mathrm{Con}} \mathbb{D}$ and $\mathbb{D} \leq_{\mathrm{Con}} \mathbb{C}$. (pp-equivalent)
- $\overline{\mathcal{C}}$ is the $\equiv_{\mathrm{m}}$-class of $\mathcal{C}$ ($\overline{\mathbb{C}}$ is the $\equiv_{\mathrm{Con}}$-class of $\mathbb{C}$).

Definition

$$\mathfrak{P}_{\mathrm{fin}} := \left(\{\overline{\mathcal{C}} \mid \mathcal{C} \textit{ is a clone over some finite set}\}; \leq_{\mathrm{m}}\right)$$
$$\mathfrak{P}_n := \left(\{\overline{\mathcal{C}} \mid \mathcal{C} \textit{ is a clone over } \{0, \dots, n-1\}\}; \leq_{\mathrm{m}}\right)$$

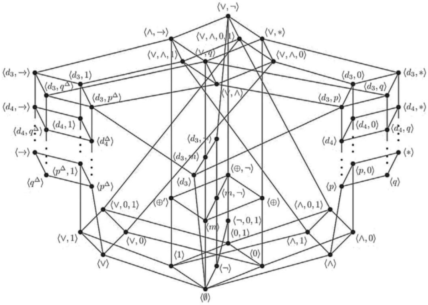


How powerful are minor-preserving maps?

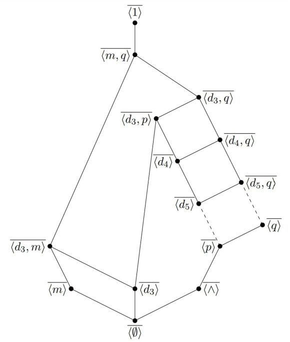


Post's lattice
(Post '41)

How powerful are minor-preserving maps?

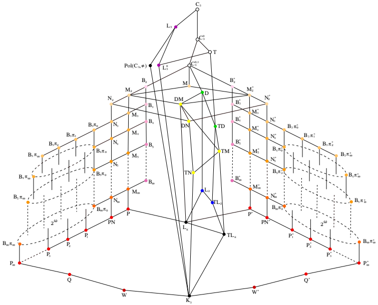


Post's lattice
(Post '41)



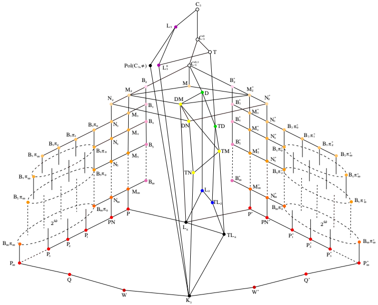
$\mathcal{P}_2$
(Bodirsky, V. 2020)

How powerful are minor-preserving maps?

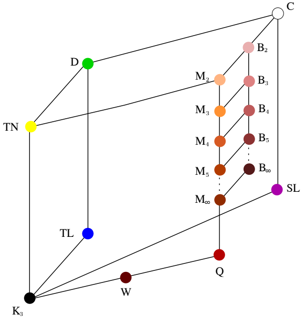


Clones of self-dual operations
(Zhuk 2015)

How powerful are minor-preserving maps?



Clones of self-dual operations
(Zhuk 2015)



Clones of self-dual operations
modulo minor-equivalence
(Bodirsky, V., Zhuk 2023)

$\mathfrak{P}_{\text{fin}}$ is a semilattice

- $\mathbb{A}$ and $\mathbb{B}$ be finite relational structures;

$\mathfrak{P}_{\text{fin}}$ is a semilattice

- $\mathbb{A}$ and $\mathbb{B}$ be finite relational structures;
- for every $f \in \text{Pol}(\mathbb{A})$, $g \in \text{Pol}(\mathbb{B})$; define an operation h on $A \times B$ $h := (f, g) \in \text{Pol}(\mathbb{A}) \times \text{Pol}(\mathbb{B})$ as follows

$$h((a_1, b_1), \ldots, (a_n, b_n)) := (f(a_1, \ldots, a_n), g(b_1, \ldots, b_n))$$

where $a_i \in A$ and $b_i \in B$ for every $i \in \{1, \ldots, n\}$.

$\mathfrak{P}_{\text{fin}}$ is a semilattice

- $\mathbb{A}$ and $\mathbb{B}$ be finite relational structures;
- for every $f \in \text{Pol}(\mathbb{A})$, $g \in \text{Pol}(\mathbb{B})$; define an operation h on $A \times B$ $h := (f, g) \in \text{Pol}(\mathbb{A}) \times \text{Pol}(\mathbb{B})$ as follows

$$h((a_1, b_1), \ldots, (a_n, b_n)) := (f(a_1, \ldots, a_n), g(b_1, \ldots, b_n))$$

  where $a_i \in A$ and $b_i \in B$ for every $i \in \{1, \ldots, n\}$.
- $\Gamma^{\mathbb{A}\otimes\mathbb{B}} := \text{Inv}(\{(f, g) \mid f \in \text{Pol}(\mathbb{A}), g \in \text{Pol}(\mathbb{B})\})$; we define

$$\mathbb{A} \otimes \mathbb{B} := (A \times B; \Gamma^{\mathbb{A}\otimes\mathbb{B}}).$$

Proposition

$\overline{\mathbb{A} \otimes \mathbb{B}}$ *is the greatest lower bound of* $\overline{\mathbb{A}}$ *and* $\overline{\mathbb{B}}$.

Are there atoms in $\mathfrak{P}_{\mathrm{fin}}$?

Are there atoms in $\mathfrak{P}_{\text{fin}}$?

Theorem

$\mathfrak{P}_{\text{fin}}$ *has no atoms.*

Are there atoms in $\mathfrak{P}_{\text{fin}}$?

Theorem

$\mathfrak{P}_{\text{fin}}$ *has no atoms.*

Sketch of the proof:

- given a finite structure $\mathbb{A}$ such that $\overline{\mathbb{A}} \neq \overline{\mathbb{K}_3}$, $(\star)$;
- show: $\exists \mathbb{B}$ finite structure such that $\overline{\mathbb{B}} <_{\text{Con}} \overline{\mathbb{A}}$ and $\overline{\mathbb{B}} \neq \overline{\mathbb{K}_3}$;

Are there atoms in $\mathfrak{P}_{\mathrm{fin}}$?

Theorem

$\mathfrak{P}_{\mathrm{fin}}$ *has no atoms.*

Sketch of the proof:

- given a finite structure $\mathbb{A}$ such that $\overline{\mathbb{A}} \neq \overline{\mathbb{K}_3}$, $(\star)$;
- show: $\exists \mathbb{B}$ finite structure such that $\overline{\mathbb{B}} <_{\mathrm{Con}} \overline{\mathbb{A}}$ and $\overline{\mathbb{B}} \neq \overline{\mathbb{K}_3}$;
- from $(\star)$ it follows that $\mathbb{A} \models c(x_1, \ldots, x_p) \approx c(x_2, \ldots, x_p, x_1)$, for some prime $p > |A|$ ($\mathbb{A} \models \Sigma_p$);

Are there atoms in $\mathfrak{P}_{\text{fin}}$?

Theorem

$\mathfrak{P}_{\text{fin}}$ *has no atoms.*

Sketch of the proof:

- given a finite structure $\mathbb{A}$ such that $\overline{\mathbb{A}} \neq \overline{\mathbb{K}_3}$, $(\star)$;
- show: $\exists \mathbb{B}$ finite structure such that $\overline{\mathbb{B}} <_{\text{Con}} \overline{\mathbb{A}}$ and $\overline{\mathbb{B}} \neq \overline{\mathbb{K}_3}$;
- from $(\star)$ it follows that $\mathbb{A} \models c(x_1, \ldots, x_p) \approx c(x_2, \ldots, x_p, x_1)$, for some prime $p > |A|$ ($\mathbb{A} \models \Sigma_p$);
- take $\mathbb{B} = \mathbb{A} \otimes \mathbb{C}_p$
  1. $\mathbb{B} \not\models \Sigma_p \implies \overline{\mathbb{B}} <_{\text{Con}} \overline{\mathbb{A}}$
  2. $\mathbb{B} \models \Sigma_q$, for some $q > p \cdot |A| \implies \overline{\mathbb{B}} \neq \overline{\mathbb{K}_3}$.

Are there atoms in $\mathfrak{P}_n$?

Where to look:

- Minimal Taylor Clones
  Barto, Brady, Bulatov, Kozik, and Zhuk (2021)

Are there atoms in $\mathfrak{P}_n$?

Where to look:

- Minimal Taylor Clones
  Barto, Brady, Bulatov, Kozik, and Zhuk (2021)
- $\overline{\mathcal{C}}$ atom in $\mathfrak{P}_n \implies \mathcal{C}$ is a minimal Taylor clone over $\{0, \ldots, n-1\}$;

Are there atoms in $\mathfrak{P}_n$?

Where to look:

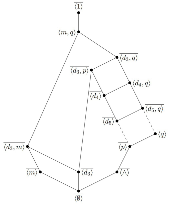
- Minimal Taylor Clones
  Barto, Brady, Bulatov, Kozik, and Zhuk (2021)
- $\overline{\mathcal{C}}$ atom in $\mathfrak{P}_n \implies \mathcal{C}$ is a minimal Taylor clone over $\{0, \ldots, n-1\}$;
- What about the other direction ($\Longleftarrow$)?

Are there atoms in $\mathfrak{P}_n$?

Where to look:

- Minimal Taylor Clones
  Barto, Brady, Bulatov, Kozik, and Zhuk (2021)
- $\overline{\mathcal{C}}$ atom in $\mathfrak{P}_n \implies \mathcal{C}$ is a minimal Taylor clone over $\{0, \ldots, n-1\}$;
- What about the other direction ($\Longleftarrow$)?
  1. n = 2 Minimal Taylor clones: $\langle \vee \rangle$, $\langle \wedge \rangle$, $\langle d_3 \rangle$, $\langle m \rangle$

     Atoms in $\mathfrak{P}_2$: $\overline{\langle \vee \rangle} = \overline{\langle \wedge \rangle}$, $\overline{\langle m \rangle}$, $\overline{\langle d_3 \rangle}$.

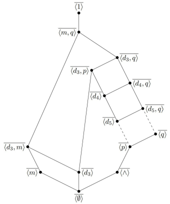


Are there atoms in $\mathfrak{P}_n$?

Where to look:

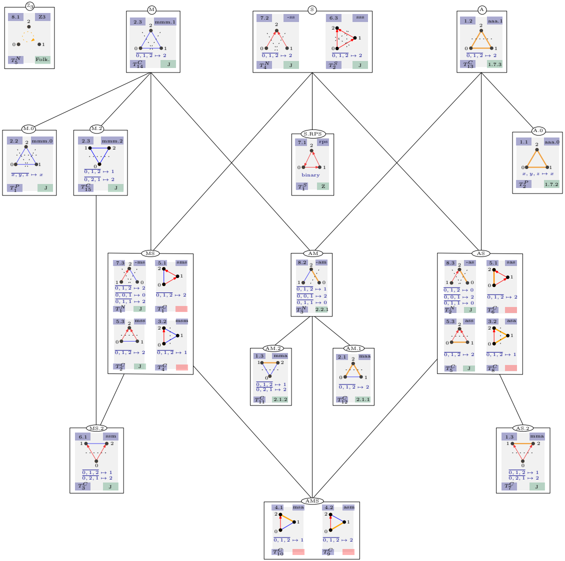
- Minimal Taylor Clones
  Barto, Brady, Bulatov, Kozik, and Zhuk (2021)
- $\overline{\mathcal{C}}$ atom in $\mathfrak{P}_n \implies \mathcal{C}$ is a minimal Taylor clone over $\{0, \ldots, n-1\}$;
- What about the other direction ($\Longleftarrow$)?
  1. n = 2 Minimal Taylor clones: $\langle\vee\rangle$, $\langle\wedge\rangle$, $\langle d_3\rangle$, $\langle m\rangle$

     Atoms in $\mathfrak{P}_2$: $\overline{\langle\vee\rangle} = \overline{\langle\wedge\rangle}$, $\overline{\langle m\rangle}$, $\overline{\langle d_3\rangle}$.



  2. n = 3 False! $\implies$ "Atoms are better than Minimal Taylor"
     (Barto, Brady, Jankovec, V., Zhuk)

Are there atoms in $\mathfrak{P}_n$?



Submaximal elements in $\mathfrak{P}_3$

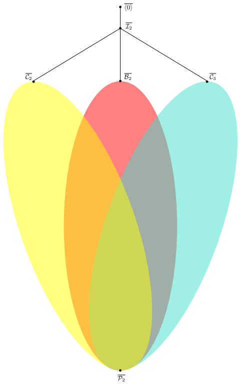
$\mathbb{C}_p$: directed cycle of length p;
$\mathbb{B}_2 = (\{0,1\}; \{(0,1),(1,0),(1,1)\})$.

Submaximal elements in $\mathfrak{P}_3$

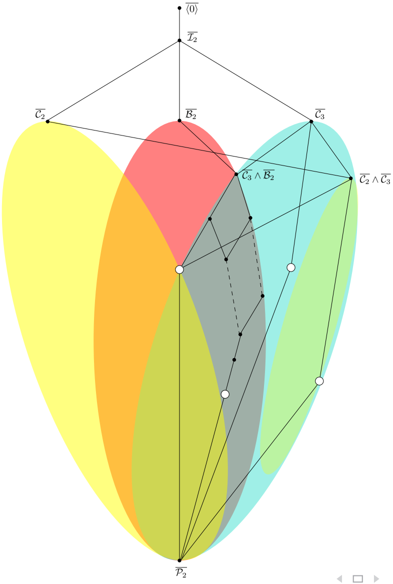
$\mathbb{C}_p$: directed cycle of length p;
$\mathbb{B}_2 = (\{0,1\}; \{(0,1),(1,0),(1,1)\})$.

Theorem (V., Zhuk)

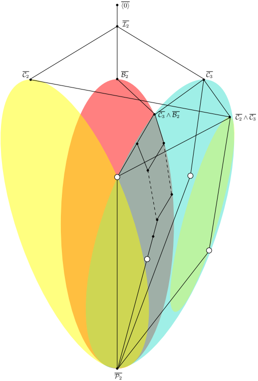
$\mathfrak{P}_3$ has exactly three submaximal elements: $\overline{\mathcal{C}_2}$, $\overline{\mathcal{C}_3}$, and $\overline{\mathcal{B}_2}$



Submaximal elements in $\mathfrak{P}_3$

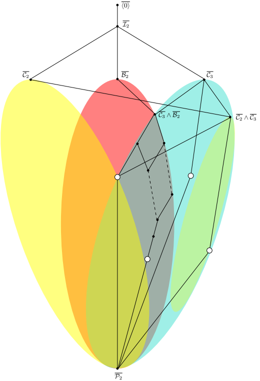


Cardinality of $\mathfrak{P}_3$



- Below $\overline{\mathcal{C}_3}$: Fully described. (Bodirsky, V., Zhuk)

Cardinality of $\mathfrak{P}_3$

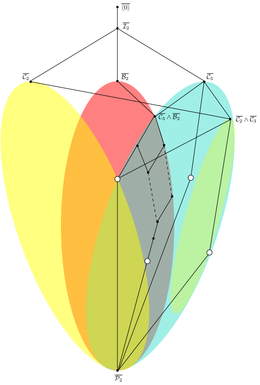


- Below $\overline{\mathcal{C}_3}$: Fully described. (Bodirsky, V., Zhuk)

Theorem (Bulatov 2001)

There are only finitely many clones on $\{0, 1, 2\}$ with a Mal'cev operation.

Cardinality of $\mathfrak{P}_3$



- Below $\overline{\mathcal{C}_3}$: Fully described. (Bodirsky, V., Zhuk)

Theorem (Bulatov 2001)

There are only finitely many clones on $\{0, 1, 2\}$ with a Mal'cev operation.

- Below $\overline{\mathcal{C}_2}$: Mild! ☺
- Below $\overline{\mathcal{B}_2}$: Wild! (potentially 2^ω elements) ☹

Ongoing and future

1. Is $\mathfrak{P}_{\text{fin}}$ a lattice?

Ongoing and future

1. Is $\mathfrak{P}_{\mathrm{fin}}$ a lattice?
2. Cardinality of $\mathfrak{P}_{\mathrm{fin}}$: We know where to look (again below $\overline{\mathbb{B}_2}$).

Theorem (Aichinger, Mayr, McKenzie 2014)

There are only countably many clones over $\{0, \ldots, n-1\}$ containing a Mal'cev operation.

Ongoing and future

1. Is $\mathfrak{P}_{\mathrm{fin}}$ a lattice?
2. Cardinality of $\mathfrak{P}_{\mathrm{fin}}$: We know where to look (again below $\overline{\mathbb{B}_2}$).

Theorem (Aichinger, Mayr, McKenzie 2014)

There are only countably many clones over $\{0, \dots, n-1\}$ containing a Mal'cev operation.

3. Mal'cev clones over $\{0, 1, 2\}$ up to minor-equivalence (Fioravanti, Rossi, V.).

Ongoing and future

1. Is $\mathfrak{P}_{\text{fin}}$ a lattice?
2. Cardinality of $\mathfrak{P}_{\text{fin}}$: We know where to look (again below $\overline{\mathbb{B}_2}$).

Theorem (Aichinger, Mayr, McKenzie 2014)

There are only countably many clones over $\{0, \ldots, n-1\}$ containing a Mal'cev operation.

3. Mal'cev clones over $\{0, 1, 2\}$ up to minor-equivalence (Fioravanti, Rossi, V.).
4. Clones "defined by binary relations"
   see D. Zhuk, PALS – 14 March 2023 (on Youtube)

European Union (ERC, POCOCOP, 101071674). Views and opinions expressed are however those of the author(s) only and do not necessarily reflect those of the European Union or the European Research Council Executive Agency. Neither the European Union nor the granting authority can be held responsible for them.