TOPOLOGIES ON ENDOMORPHISM MONOIDS OF GENERIC STRUCTURES

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HO!

CONFERENCE ON GENERIC STRUCTURES

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PART I : MOTIVATION & RESULTS

- ALGEBRAIC - TOPOLOGICAL STRUCTURES
- AUTOMORPHISM GROUPS, ENDOMORPHISM MONOIDS, POLYMORPHISM CLONES
- RECONSTRUCTION NOTIONS
- THE GRAND TABLE OF RESULTS

PART II : METHODS

- THE ZARISKI TOPOLOGY
- HOMOMORPHISM GLUING & HOMOGENEITY

MANY MATHEMATICAL OBJECTS CARRY:

- ALGEBRAIC STRUCTURE
- TOPOLOGICAL STRUCTURE

COMPATIBLE: ALGEBRAIC OPERATIONS CONTINUOUS.

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EXAMPLES

- GROUP $(\mathbb{R},+)$
- GROUP $(\mathbb{R}^2,+)$
- VECTOR SPACES OVER $\mathbb{R}/\mathbb{C}/\mathbb{K}$: SUBSETS OF $\mathbb{R}^I / \mathbb{C}^I / \mathbb{K}^I$ PRODUCT TOPOLOGIES

  $\mathbb{K}$ DISCRETE $\Rightarrow$ TOPOLOGY OF POINTWISE CONVERGENCE
- IN PARTICULAR: VECTOR SPACE $\mathbb{R}^n$
- GROUP $\mathrm{Aut}(\mathbb{Q},<) \subseteq \mathbb{Q}^{\mathbb{Q}}$: PRODUCT TOPOLOGIES
- IN GENERAL: GROUP $\mathrm{Aut}(\mathbb{A})$, $\mathbb{A}$ FIRST-ORDER STRUCTURE: POINTWISE CONVERGENCE

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- SEMIGROUPS (MONOIDS): $\mathrm{End}(\mathbb{A})$ ENDOMORPHISM MONOID

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- CLONE $\mathrm{Pol}(\mathbb{A}) = \bigcup_{n \in \omega} \{ f : \mathbb{A}^n \to \mathbb{A} \mid f \text{ PRESERVES } \mathbb{A} \}$

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  $\mathrm{Pol}\,\mathbb{A} \cap \mathbb{A}^{\mathbb{A}^n}$ POINTWISE CONV., CLOPEN

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  $\mathrm{Pol}\,\mathbb{A} \cap \mathbb{A}^{\mathbb{A}^n}$ POINTWISE CONV., CLOPEN
- ALGEBRA $(A; (f_i)_{i \in I})$: COMPOSITION + POINTWISE CONVERGENCE.

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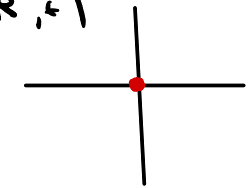

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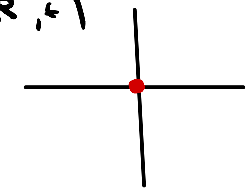

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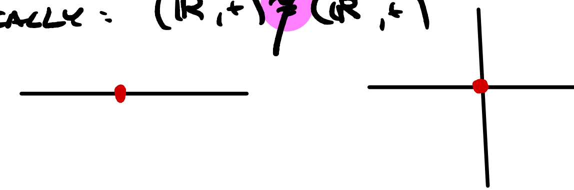

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QUESTION (MARKOV 44)

IS THERE AN INFINITE NON-TOPOLOGIZABLE GROUP?
(ONLY HAUSDORFF TOPOLOGY = DISCRETE)

(YES: SHELAH (CH), HESSE, OLSHANSKII '80)

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QUESTION IS THERE A UNIQUE ... TOPOLOGY?

... E.G.
- T_2 / HAUSDORFF
- SEPARABLE
- POLISH = SEPARABLE + COMPLETELY METRIZABLE

GROUPS

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- **EXAMPLE** $(\mathbb{R},+) \cong (\mathbb{R}^2,+)$, $(\mathbb{R},+) \not\cong (\mathbb{R}^2,+)$
- <u>ON THE OTHER HAND</u>: CONSISTENT WITH ZF THAT ANY POLISH GROUP HAS UNIQUE POLISH TOPOLOGY (PETTIS, SOLOVAY, SHELAH)

:-)

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ANSWER <u>NO</u>:

- EVERY T_1 TOPOLOGY $\supseteq$ PW (GAUGHAN '70s)
- $\mathcal{J} \subseteq \mathcal{J}'$, BOTH POLISH $\Rightarrow \mathcal{J} = \mathcal{J}'$

$\Rightarrow$ PW UNIQUE POLISH

(EVEN SEPARABLE)

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UNIQUE POLISH TOPOLOGY:

- ISOMETRY GROUP OF THE URYSOHN SPACE / SPHERE (SABOK '13)
- HOMEOMORPHISM GROUPS OF $[0,1]^{\mathbb{N}}$, $2^{\mathbb{N}}$ (KALLMAN '80s)
- $\mathrm{Aut}(\mathbb{Q},<)$ (HODGES, HODKINSON, LASCAR, SHELAH '93 / SOLECKI + ROSENDAL '07)
- $\mathrm{Aut}(G)$ G … RANDOM GRAPH (HRUSHOVSKI '92 / KECHRIS + ROSENDAL '07)

SEMIGROUPS / CLONES

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UNIQUE POLISH TOPOLOGY:

- $\mathbb{N}^{\mathbb{N}}$
- $\bigcup_{n \in \omega} \mathbb{N}^{\mathbb{N}^n}$

(ELLIOTT + JONUŠAS + MESYAN + MITCHELL + MORAYNE + PÉRESSE '18)

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INFINITELY MANY POLISH TOPOLOGIES:

$\mathrm{Inj}(\mathbb{N}) := \{f : \mathbb{N} \to \mathbb{N} \mid f \text{ injective}\}$ $\left(= \overline{\mathrm{Aut}(\mathbb{N}, \neq)}^{pw} \right)$

SMALLEST: pw : GENERATED BY $U_{x,y} = \{f \in \mathbb{N}^{\mathbb{N}} \mid f(x) = y\}$ $(x, y \in \mathbb{N})$

LARGEST: pw + $N_x := \{f \in \mathbb{N}^{\mathbb{N}} \mid x \notin f(\mathbb{N})\}$ $(x \in \mathbb{N})$

$M_n := \{f \in \mathbb{N}^{\mathbb{N}} \mid |\mathbb{N} \setminus f(\mathbb{N})| = n\}$ $(n \leq \omega)$

IN BETWEEN: RESTRICT n

(NO CLASSIFICATION)

WHAT FOR?

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INSIGHT INTO RELATIONSHIP BETWEEN ALGEBRAIC & TOPOLOGICAL STRUCTURE

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PRACTICAL USE ALGEBRAIC OBJECT AS INVARIANT:

14

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INSIGHT INTO RELATIONSHIP BETWEEN ALGEBRAIC & TOPOLOGICAL STRUCTURE

PRACTICAL USE ALGEBRAIC OBJECT AS INVARIANT:

EXAMPLES LET $\mathbb{A}, \mathbb{B}$ BE COUNTABLE <u>ω-CATEGORICAL</u>

- $\mathbb{A}, \mathbb{B}$ FIRST-ORDER BI-INTERPRETABLE $\Leftrightarrow$ $\mathrm{Aut}(\mathbb{A}) \cong \mathrm{Aut}(\mathbb{B})$ (AHLBRANDT ZIEGLER '80s)

  IF $\mathbb{A}$ HAS UNIQUE POLISH TOP.: $\Leftrightarrow$ $\mathrm{Aut}(\mathbb{A}) \cong \mathrm{Aut}(\mathbb{B})$ COQUAND

- $\mathbb{A}, \mathbb{B}$ EXISTENTIAL-POSITIVE BI-INT. $\Leftrightarrow$ $\mathrm{End}(\mathbb{A}) \cong \mathrm{End}(\mathbb{B})$ (BODIRSKY + JUNKER '08)

- $\mathbb{A}, \mathbb{B}$ PRIMITIVE-POSITIVE BI-INT. $\Leftrightarrow$ $\mathrm{Pol}(\mathbb{A}) \cong \mathrm{Pol}(\mathbb{B})$

(BODIRSKY + P. '11)

$\boxed{\text{VARIANT}}$ AUTOMATIC CONTINUITY AC

$\mathrm{Aut}(\mathbb{A})$ HAS AC $:\Leftrightarrow$ $\forall \varphi : \mathrm{Aut}(\mathbb{A}) \to \mathrm{Sym}(\mathbb{N})$ (φ CONTINUOUS)

$\mathrm{End}(\mathbb{A})$ HAS AC $:\Leftrightarrow$ $\mathrm{End}(\mathbb{A}) \to \mathbb{N}^{\mathbb{N}}$

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MEANING FOR ω-CATEGORICAL $\mathbb{A}$:

$\mathbb{B}$ HAS FO-INTERPRETATION IN $\mathbb{A}$ $\Leftrightarrow$ $\mathrm{Aut}(\mathbb{A}) \to \mathrm{Aut}(\mathbb{B})$

EP- $\qquad \mathrm{End}(\mathbb{A}) \to \mathrm{End}(\mathbb{B})$

PP- $\qquad \mathrm{Pol}(\mathbb{A}) \to \mathrm{Pol}(\mathbb{B})$

(MORE OR LESS)

VARIANT AUTOMATIC CONTINUITY AC

$Aut(\mathbb{A})$ HAS AC :$\Leftrightarrow$ $\forall \varphi : Aut(\mathbb{A}) \to Sym(\mathbb{N})$ (φ CONTINUOUS)

$End(\mathbb{A})$ HAS AC :$\Leftrightarrow$ $End(\mathbb{A}) \to \mathbb{N}^{\mathbb{N}}$

$Pol(\mathbb{A})$ HAS AC :$\Leftrightarrow$ $Pol(\mathbb{A}) \to \bigcup_{n \in \omega} \mathbb{N}^{\mathbb{N}^n}$

MEANING FOR ω- CATEGORICAL $\mathbb{A}$:

$\mathbb{B}$ HAS FO-INTERPRETATION IN $\mathbb{A}$ $\Leftrightarrow$ $Aut(\mathbb{A}) \to Aut(\mathbb{B})$

EP- $End(\mathbb{A}) \to End(\mathbb{B})$

PP - $Pol(\mathbb{A}) \to Pol(\mathbb{B})$

(MORE OR LESS)

EXAMPLE : CAN $\mathbb{A}$ PP-INTERPRET ALL FINITE STRUCTURES ?

VARIANT AUTOMATIC CONTINUITY AC

$\mathrm{Aut}(\mathbb{A})$ HAS AC :⇔ $\forall \varphi: \mathrm{Aut}(\mathbb{A}) \to \mathrm{Sym}(\mathbb{N})$ (φ CONTINUOUS)

$\mathrm{End}(\mathbb{A})$ HAS AC :⇔ $\mathrm{End}(\mathbb{A}) \to \mathbb{N}^{\mathbb{N}}$

$\mathrm{Pol}(\mathbb{A})$ HAS AC :⇔ $\mathrm{Pol}(\mathbb{A}) \to \bigcup_{n\in\omega} \mathbb{N}^{\mathbb{N}^n}$

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(MORE OR LESS)

EXAMPLE: CAN $\mathbb{A}$ PP-INTERPRET ALL FINITE STRUCTURES?

<u>THM</u>: $\mathrm{Pol}(\mathbb{A}) \not\to \mathrm{Pol}(\text{ANY FINITE}) \Leftrightarrow \exists e, f, s$

$$\mathrm{Pol}(\mathbb{A}) \models e \circ s(x,y,x,z,y,z) = f \circ s(y,x,z,x,z,y)$$

(BARTO + P. '16, NOT TRUE AS STATED)

RECONSTRUCTING THE TOPOLOGY OF $\mathrm{Aut}(\mathbb{A})$, $\mathrm{End}(\mathbb{A})$, $\mathrm{Pol}(\mathbb{A})$

$\mathbb{A}$... COUNTABLE ω-CATEGORICAL

THREE NOTIONS:

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UP: PW IS UNIQUE POLISH TOPOLOGY ON $\underbrace{\mathrm{Aut}(\mathbb{A})}_{\text{HARD TO FAIL}} / \underbrace{\mathrm{End}(\mathbb{A}) / \mathrm{Pol}(\mathbb{A})}_{\text{FAILS EASILY}}$

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THREE NOTIONS:

HARD TO FAIL / FAILS EASILY

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AC: EVERY $\varphi: \mathrm{Aut}(\mathbb{A}) \to \mathrm{Sym}(\mathbb{N})$ } HARD TO FAIL

IS CONTINUOUS

FAILS EASILY {

$\mathrm{End}(\mathbb{A}) \to \mathbb{N}^{\mathbb{N}}$

$\mathrm{Pol}(\mathbb{A}) \to \bigcup_{n\in\omega} \mathbb{N}^{\mathbb{N}^n}$

(AUTOMATIC CONTINUITY)

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$\mathbb{A}$… COUNTABLE ω-CATEGORICAL

THREE NOTIONS:

HARD TO FAIL

FAILS EASILY

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$\mathrm{End}(\mathbb{A}) \to \mathbb{N}^{\mathbb{N}}$
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(**A**UTOMATIC **C**ONTINUITY)

AH: EVERY ISOMORPHISM
$\varphi: \mathrm{Aut}(\mathbb{A}) \to \mathrm{Aut}(\mathbb{B})$
$\mathrm{End}(\mathbb{A}) \to \mathrm{End}(\mathbb{B})$
$\mathrm{Pol}(\mathbb{A}) \to \mathrm{Pol}(\mathbb{B})$

IS A HOMEOMORPHISM

(**A**UTOMATIC **H**OMEOMORPHICITY)

HARD TO FAIL

EVANS + HEWITT '90
+ BODIRSKY + KOMPATSCHER + P. '15

RECONSTRUCTING THE TOPOLOGY OF $\mathrm{Aut}(\mathbb{A})$, $\mathrm{End}(\mathbb{A})$, $\mathrm{Pol}(\mathbb{A})$

$\mathbb{A}$... COUNTABLE ω-CATEGORICAL

THREE NOTIONS:

HARD TO FAIL (Aut) — FAILS EASILY (End / Pol)

UP: PW IS UNIQUE POLISH TOPOLOGY ON $\mathrm{Aut}(\mathbb{A})$ / $\mathrm{End}(\mathbb{A})$ / $\mathrm{Pol}(\mathbb{A})$

AC: EVERY $\varphi: \mathrm{Aut}(\mathbb{A}) \to \mathrm{Sym}(\mathbb{N})$ } HARD TO FAIL IS CONTINUOUS

FAILS EASILY { $\mathrm{End}(\mathbb{A}) \to \mathbb{N}^{\mathbb{N}}$, $\mathrm{Pol}(\mathbb{A}) \to \bigcup_{n \in \omega} \mathbb{N}^{\mathbb{N}^n}$

(AUTOMATIC CONTINUITY)

AH: EVERY ISOMORPHISM $\varphi: \mathrm{Aut}(\mathbb{A}) \to \mathrm{Aut}(\mathbb{B})$, $\mathrm{End}(\mathbb{A}) \to \mathrm{End}(\mathbb{B})$, $\mathrm{Pol}(\mathbb{A}) \to \mathrm{Pol}(\mathbb{B})$ IS A HOMEOMORPHISM

HARD TO FAIL

(AUTOMATIC HOMEOMORPHICITY)

EVANS + HEWITT '90 + BODIRSKY + KOMPATSCHER + P. '15

AR: EVERY ISOMORPHISM φ IS INDUCED BY BIJECTION $f: \mathbb{A} \to \mathbb{B}$

(ACTION RECONSTRUCTION)

$$\begin{array}{ccccc} & & & & \mathrm{AC} \\ & & & \swarrow & \\ \mathrm{UP} & \longrightarrow & \mathrm{AH} & & \\ & & & \nwarrow & \\ & & & & \mathrm{AR} \end{array}$$

FOR GROUPS (OPEN FOR MONOIDS)

AR USUALLY RESTRICTED TO $\mathbb{B}$ SATISFYING CONDITIONS: NO ALGEBRAICITY

THE GRAND TABLE OF RESULTS

(INCOMPLETE + PROBABLY INCORRECT)

	UNIQUE POLISH (UP)	AUTOMATIC CONT. (AC)	ACTION REC. (AR)	AUTOMATIC HOMEO (AH)
Aut	$(\mathbb{N},=)$ Gaughan '70s $(\mathbb{Q},<)$ Rosendahl + Solecki '07 $\mathbb{G}$ Kechris + Rosendal '07	$(\mathbb{N},=)$ Dixon, Neumann, Thomas '86, Semmes '81 $(\mathbb{Q},<)$, $\mathbb{B}$ Truss '89 $\mathbb{G}$, ω-stable Hodges + Hodkinson + Lascar + Shelah '80s $\mathbb{H}_n$ Herwig '80s	$(\mathbb{Q},<)$, $\mathbb{G}$, $\mathbb{H}_n$, $\mathbb{P}$, $\mathbb{T}$ Rubin '94 Hypergraphs, Henson digraphs Barbina + Macpherson '07	$(\mathbb{Q},<)$, $\mathbb{G}$, $\mathbb{H}_n$, $\mathbb{P}$, $\mathbb{T}$ Rubin '94
End	$\mathbb{G}$, $\mathbb{D}$, $\mathbb{E}$, $(\mathbb{P},<)$, $(\mathbb{P},\leq)$ Elliott + Jonušas + Mitchell + Péresse + P. '21 $(\mathbb{Q},\leq)$ Schneider + P. '23 $(\mathbb{N},=)$ EJMMMP '20	$\mathbb{G}$, $\mathbb{D}$, $\mathbb{E}$ Elliott + Jonušas + Mitchell + Péresse + P. '21 $(\mathbb{N},=)$ EJMMMP '20	$(\mathbb{Q},<)$, $(\mathbb{P},<)$, $\mathbb{G}$, $\mathbb{H}_n$ Behrisch + Vargas-García '19	$(\mathbb{N},\neq)$, $\mathbb{G}$ / $\mathbb{T}$ Bodirsky + P. + Pongracz '13 $(\mathbb{Q},<)$, $(\mathbb{Q},\leq)$ Behrisch + Truss + Vargas-García '17
Pol	$\mathbb{G}$, $\mathbb{D}$, $(\mathbb{P},<)$, $(\mathbb{P},\leq)$, $\mathbb{E}$ Elliott + Jonušas + Mitchell + Péresse + P. '21 $(\mathbb{N},=)$ Elliott + Jonušas + Mesyan + Mitchell + Morayne + Péresse '20	$\mathbb{G}$, $\mathbb{D}$, $\mathbb{E}$ Elliott + Jonušas + Mitchell + Péresse + P. '21 $(\mathbb{N},=)$ EJMMMP '20	$(\mathbb{Q},<)$, $(\mathbb{P},<)$, $\mathbb{G}$, $\mathbb{H}_n$ Behrisch + Vargas-García '19	$\supseteq \mathbb{N}^{\mathbb{N}}$, $\mathbb{G}$ Bodirsky + P. + Pongracz '13 $(\mathbb{P},\leq)$, $(\mathbb{P},<)$ Pech + Pech '18 $(\mathbb{Q},\leq)$ Behrisch + Truss + Vargas-García '17 $(\mathbb{Q},<)$ Behrisch + Vargas-García '19

$\mathbb{G}$ ... RANDOM GRAPH

$\mathbb{P}$ ... RANDOM POSET

$\mathbb{T}$ ... RANDOM TOURNAMENT

$\mathbb{D}$ ... RANDOM DIGRAPH

$\mathbb{E}$ ... RANDOM EQUIVALENCE RELATION

$\mathbb{H}_n$ ... RANDOM K_n-FREE GRAPH

$\mathbb{B}$ ... COUNTABLE ATOMLESS BOOLEAN ALG

ω-CAT

NO ALGEBRAICITY

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ZARISKI TOPOLOGY

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- CLOSED SETS: ALGEBRAICALLY GIVEN BY $\{s : a_1 \circ s \circ a_2 \circ \dots \circ a_n = b_1 \circ s \circ \dots \circ s \circ b_m\}$
  (AS OPPOSED TO PW)

- ANY T_1 SEMIGROUP TOPOLOGY $\supseteq$ ZARISKI

- ZARISKI NEED NOT BE T_2 OR A SEMIGROUP TOPOLOGY!

ZARISKI TOPOLOGY

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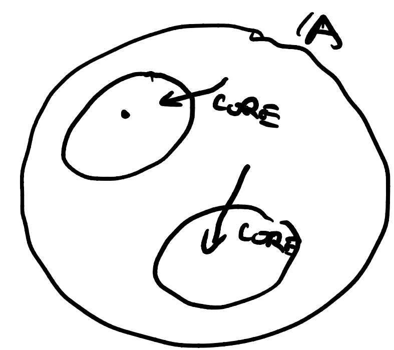
- ZARISKI NEED NOT BE T_2 OR A SEMIGROUP TOPOLOGY!

THEOREM (P. + SCHINDLER '23)

$\mathbb{A}$ ω-CATEGORICAL, NO ALGEBRAICITY

- EVERY ELEMENT CONTAINED IN CORE
- CORE FINITE OR NO ALGEBRAICITY

$\Rightarrow$ PW = ZARISKI ON $\mathrm{End}(\mathbb{A})$



IN PARTICULAR WHEN $\mathrm{End}(\mathbb{A}) = \overline{\mathrm{Aut}\,\mathbb{A}}$

ZARISKI TOPOLOGY

- CLOSED SETS: ALGEBRAICALLY GIVEN BY $\{s : a_1 \circ s \circ a_2 \circ \dots \circ a_n = b_1 \circ s \circ \dots \circ s \circ b_m\}$
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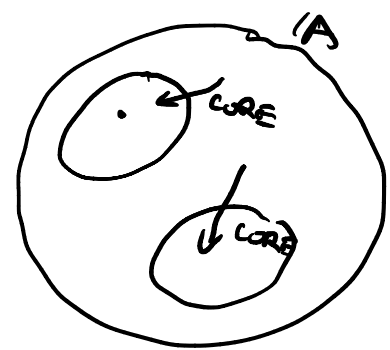

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PW $\neq$ ZARISKI:
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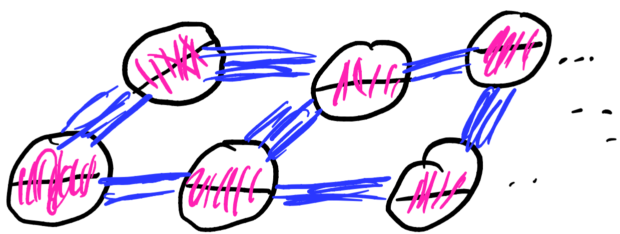

EXCLUDING FINER TOPOLOGIES

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DEFINITION

S topological semigroup, $A \subseteq S$ subsemigroup.

S has property X with respect to A

$:\Longleftrightarrow$

$\forall s \in S \ \exists f_s, g_s \in S \ \exists \alpha_s \in A$:

$$s = g_s \circ \alpha_s \circ f_s$$

AND

$\forall$ B neighbourhood of α_s :

$g_s \circ (B \cap A) \circ f_s$ is neighbourhood of s.

$$s = g_s \cdot \alpha_s \cdot f_s$$

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THEOREM (Elliott + Jonušas + Mesyan + Mitchell + Morayne + Péresse '19)

S semigroup, Polish topology $\mathcal{T}$ on S.

$A \leq S$ subsemigroup, property X.

- If A is Polish subgroup $\Rightarrow \mathcal{T}$ is a maximal Polish topology on S.
- If A has AC $\Rightarrow S$ has AC.

EXCLUDING FINER TOPOLOGIES

DEFINITION

S TOPOLOGICAL SEMIGROUP,
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$\forall s \in S \ \exists f_s, g_s \in S \ \exists \alpha_s \in A$:

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AND

$\forall B$ NEIGHBOURHOOD OF α_s :
$g_s \circ (B \cap A) \circ f_s$ IS NEIGHBOURHOOD OF s.

$$s = g_s \circ \alpha_s \circ f_s$$

THEOREM (ELLIOTT + JONUŠAS + MESYAN + MITCHELL + MORAYNE + PÉRESSE '19)

S SEMIGROUP, POLISH TOPOLOGY $\mathcal{J}$ ON S.
$A \leq S$ SUBSEMIGROUP, PROPERTY X.

- IF A IS POLISH SUBGROUP
  $\Rightarrow \mathcal{J}$ IS A MAXIMAL POLISH TOPOLOGY ON S.
- IF A HAS AC
  $\Rightarrow S$ HAS AC.

COROLLARY

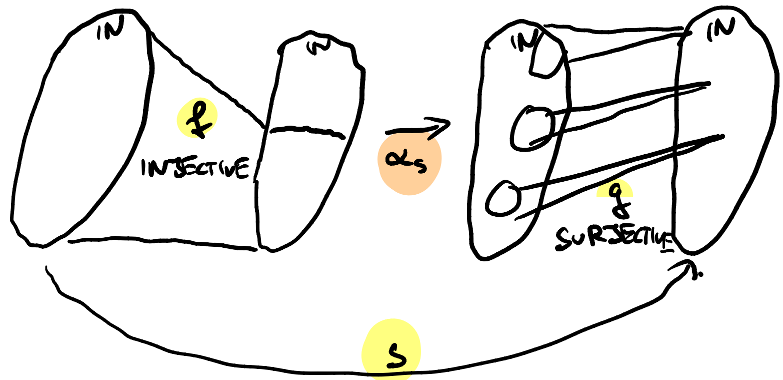
End$(\mathbb{A})$ HAS PROPERTY X WRT Aut$(\mathbb{A})$
$\Rightarrow$ PW MAXIMAL POLISH TOPOLOGY.

EXAMPLE

$\mathbb{A} = (\mathbb{N}, =)$: $\mathrm{End}(\mathbb{A}) = \mathbb{N}^{\mathbb{N}}$, $\mathrm{Aut}(\mathbb{A}) = \mathrm{Sym}(\mathbb{N})$

PROPERTY X :

$s \in \mathbb{N}^{\mathbb{N}}$



$s = g \circ \alpha_s \circ f$

SURJECTIVE INFINITE CLASSES

INJECTIVE CO-INFINITE RANGE

$\Rightarrow$ PW MAXIMAL POLISH TOP. ON $\mathbb{N}^{\mathbb{N}}$

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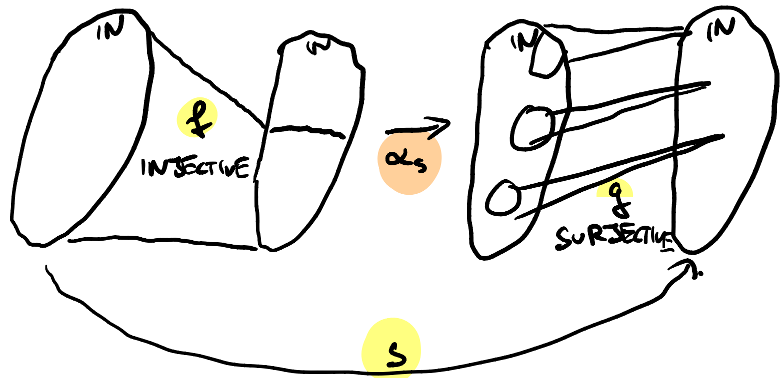

$s = g \circ \alpha_s \circ f$

SURJECTIVE INFINITE CLASSES (for g)

INJECTIVE CO-INFINITE RANGE (for f)

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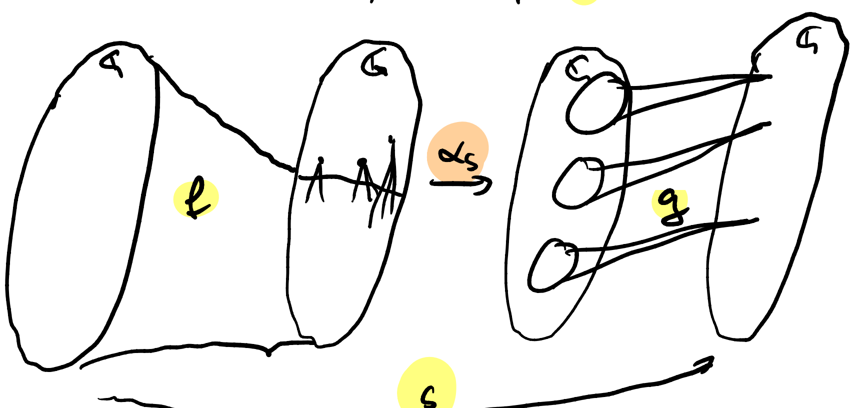
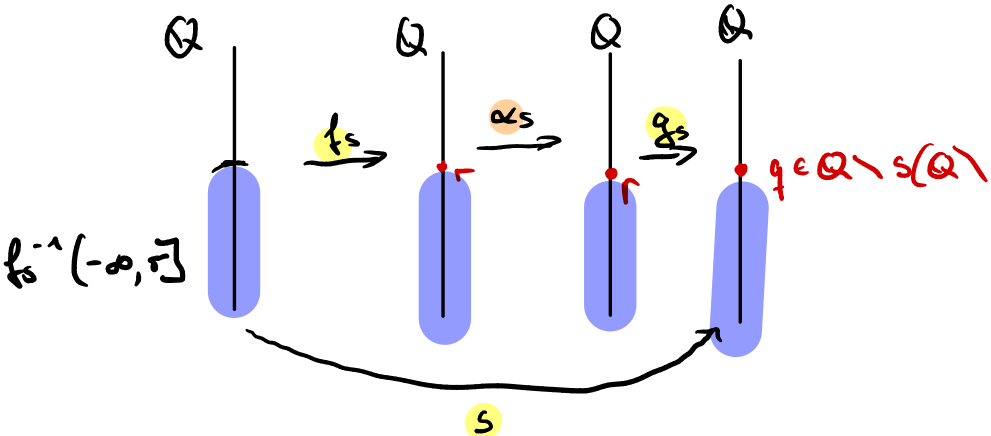
EXAMPLE $\mathbb{A} = G$ ... RANDOM GRAPH , $s \in \mathrm{End}\, G$



- g SURJECTIVE, "GENERIC"
- f EMBEDDING, GENERIC CO-IMAGE, PARAMETERS OF CO-IMAGE DO NOT DEFINE NEW TYPES IN THE IMAGE.
- α_s : BACK-AND-FORTH

NON-EXAMPLE

$\mathbb{A} = (\mathbb{Q}, \leq)$



1. TAKE s NON-SURJECTIVE, $q \notin s(\mathbb{Q})$
2. g_s HAS TO BE SURJECTIVE

   $\Rightarrow \exists p \quad g_s(p) = q$

   $\exists r \quad \alpha_s(r) = p$

3. ANY $t \in g_s \circ \alpha_s \circ f_s$ MUST SEND BELOW q

   $\uparrow$

   $r \mapsto p$

4. NO NEIGHBOURHOOD OF s IS THAT RESTRICTIVE!

SUFFICIENT CONDITIONS FOR PROPERTY X

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- IA HOMOMORPHISM-HOMOGENEOUS :$\Longleftrightarrow$ EVERY FINITE PARTIAL ENDOMORPHISM EXTENDS TO GLOBAL ENDOMORPHISM

  ⌜NON-EXAMPLE: △-FREE GRAPH; EXAMPLE: RANDOM GRAPH⌟

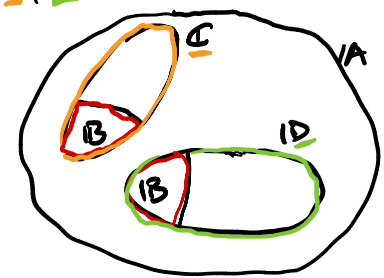
SUFFICIENT CONDITIONS FOR PROPERTY X

- $\mathbb{A}$ HOMOMORPHISM-HOMOGENEOUS $:\Longleftrightarrow$ EVERY FINITE PARTIAL ENDOMORPHISM EXTENDS TO GLOBAL ENDOMORPHISM

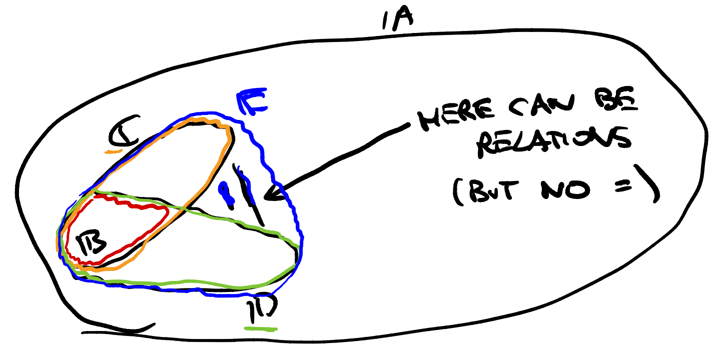
  [NON-EXAMPLE: $\triangle$-FREE GRAPH; EXAMPLE: RANDOM GRAPH]

- $\mathbb{A}$ HAS STRONG AMALGAMATION PROPERTY WITH HOMOMORPHISM GLUING $:\Longleftrightarrow$

$\forall \mathbb{B}, \mathbb{C}, \mathbb{D}$ FINITE IN $\mathbb{A}$



$\exists \mathbb{E}$ IN $\mathbb{A}$:



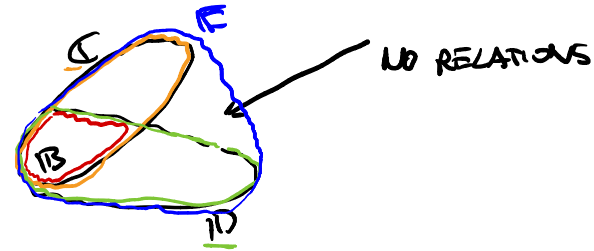
SUCH THAT $\forall\ g: \mathbb{E} \to \mathbb{F}$,

$g|_{\mathbb{C}}$ HOMO, $g|_{\mathbb{D}}$ HOMO

$\Rightarrow g$ HOMO

> "$\mathbb{E}$ IS STRONG AMALGAM WITH SMALLEST RELATIONS"

EXAMPLES



- FREE AMALGAMATION $\Rightarrow$ SAP + HG
- IN PARTICULAR: RANDOM GRAPH, DIGRAPH, ... HAVE SAP + HG
- RANDOM POSET $(\mathbb{P}, \leq)$, $(\mathbb{P}, <)$ HAVE SAP + HG
- EQUIVALENCE REL. $\mathbb{E}$ HAS SAP + HG
- $(\mathbb{Q}, \leq)$, $(\mathbb{Q}, <)$, $\mathbb{T}$ RANDOM TOURNAMENT: NO HG !

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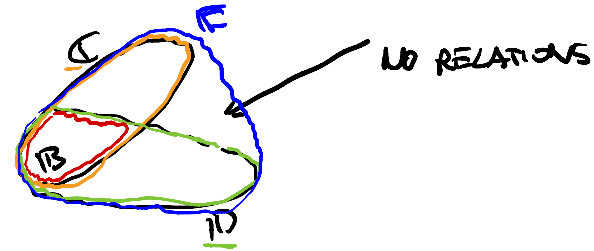

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THEOREM (ELLIOTT + JONUŠAS + MITCHELL + PÉRESSE + P. '21)

LET $\mathbb{A}$ BE HOMOGENEOUS, HOMOMORPHISM-HOMOGENEOUS, SAP + HG.

$\Rightarrow$ $\mathrm{End}(\mathbb{A})$ HAS PROPERTY X WRT $\mathrm{Aut}(\mathbb{A})$,

PW IS A MAXIMAL POLISH TOPOLOGY.

COROLLARY

UNIQUE POLISH TOP:

End / Pol OF

- RANDOM GRAPH
- DIGRAPH
- EQUIVALENCE
- POSET $\leq$
- POSET $<$
- EMPTY STRUCTURE

AUTOMATIC CONTINUITY:

End / Pol OF

- RANDOM GRAPH
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THEOREM (P. + SCHINDLER '23)

End $(\mathbb{Q}, \leq)$ HAS UNIQUE POLISH TOPOLOGY.

← NEEDS BAIRE CATEGORY THM

COROLLARY

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OPEN PROBLEMS

- THE DUAL OF THE GENERIC EQUIVALENCE RELATION?
- IS PW ALWAYS THE COARSEST T_2 TOPOLOGY?

Thank you!