Valued Constraint Satisfaction Problem and Resilience in Database Theory

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Resilience of queries

Database: a relational structure $\mathfrak{A}$

Conjunctive query: a primitive positive formula q, i.e.
$\exists y_1, \ldots, y_l \, (\psi_1 \wedge \cdots \wedge \psi_m)$, where ψ_i are atomic

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Fixed conjunctive query q.
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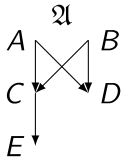
with respect to $\mathfrak{A}$ is 1 – remove (C, E).



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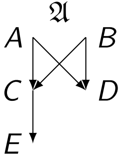

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Goal: Classify complexity of resilience for all q.

Constraint satisfaction

Fixed τ-structure $\mathfrak{A}$ (τ – finite relational signature)
Input: list of atomic τ-formulas (constraints)
Output:

- **CSP**: Decide whether there is a solution that satisfies all constraints.
- **MaxCSP**: Find the maximal number of constraints that can be satisfied at once.
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Observation: VCSP generalizes CSP and MaxCSP.
Proof: Model the tuples in relations with cost 0 and outside with cost 1 (for MaxCSP) or ∞ (for CSP).

Focus on VCSP

A valued structure Γ consists of:

- (countable) domain D
- (finite, relational) signature τ
- for each $R \in \tau$ of arity k, a function $R^{\Gamma}\colon D^k \to \mathbb{Q} \cup \{\infty\}$

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Definition (VCSP(Γ))

Input: $u \in \mathbb{Q}$, an expression

$$\phi(x_1, \ldots, x_n) = \sum_i \psi_i,$$

where each ψ_i is an atomic τ-formula

Question: Is

$$\inf_{\bar{a} \in D^n} \phi(\bar{a}) \leq u \text{ in } \Gamma?$$

Max-Cut as a VCSP

Example:

Input: $G = (V, E)$ – finite directed graph

Goal: Find a partition $A \cup B$ of V such that $E \cap (A \times B)$ is maximal.

Equivalently: $E \cap (A^2 \cup B^2 \cup B \times A)$ is minimal.

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Let Γ_{MC} be a valued structure where:

- $D = \{0, 1\}$
- $\tau = \{R\}$, R binary

$$R(x, y) = \begin{cases} 0 \text{ if } x = 0 \text{ and } y = 1 \\ 1 \text{ otherwise} \end{cases}$$

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Take vertices of G as variables. The size of a maximal cut of G is

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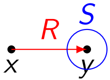
every instance of VCSP(Γ_{MC}) corresponds to a digraph

$\rightsquigarrow$ VCSP(Γ_{MC}) is the Max-Cut problem (NP-hard)

Homomorphism duality

Example (canonical structure): $\exists x, y(R(x,y) \wedge S(y)) \rightsquigarrow$ 

For a query q, take its canonical structure $\mathfrak{Q}$.
Search for a structure $\mathfrak{B}_q$ such that for every finite $\mathfrak{A}$:

$$\mathfrak{A} \not\models q \Leftrightarrow \mathfrak{Q} \not\to \mathfrak{A} \Leftrightarrow \mathfrak{A} \to \mathfrak{B}_q$$

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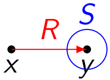
Example (canonical structure): $\exists x, y(R(x,y) \wedge S(y)) \rightsquigarrow$ 

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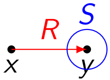
Example (canonical structure): $\exists x, y(R(x,y) \wedge S(y)) \rightsquigarrow$ 

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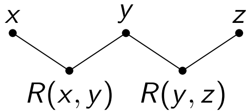
$\rightsquigarrow$ existence of $\mathfrak{B}_q$ enables studying resilience of q using the results about (valued) constraint satisfaction problems

Existence of dual structures

Example (incidence graph): $q := \exists x, y, z(R(x, y) \wedge R(y, z))$

$I(q)$:



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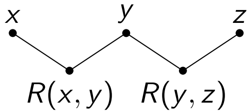

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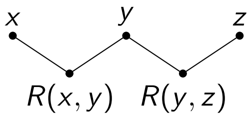

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Theorem (Cherlin, Shelah, Shi ('99))

If $I(q)$ is connected, then q has a countable dual $\mathfrak{B}_q$, which can be chosen so that $\mathrm{Aut}(\mathfrak{B}_q)$ is oligomorphic.

oligomorphic – countable domain B_q and the action of $\mathrm{Aut}(\mathfrak{B}_q)$ on B_q^n has finitely many orbits for every $n \geq 1$

Connection of resilience and VCSPs

query q with $I(q)$ connected (WLOG) $\rightsquigarrow$ obtain the dual structure $\mathfrak{B}_q$ $\rightsquigarrow$ turn it into a valued structure Γ_q with cost functions taking values 0 and 1

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Remark: We have to consider bag databases – a database $\mathfrak{A}$ might contain a tuple with multiplicity > 1 (differs from the original setting).

Example: Input $R(x, y) + R(x, y)$ for $\mathrm{VCSP}(\Gamma)$ corresponds to a database with multiplicity 2 for $R(x, y)$.

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$\mathfrak{Q} =$  $\not\to G \;\Leftrightarrow\; G \to$  $= \mathfrak{B}_q$

$\mathfrak{B}_q \rightsquigarrow \Gamma_{\mathrm{MC}} = (\{0, 1\}; R)$
Resilience of $q = \mathrm{VCSP}(\Gamma_{\mathrm{MC}}) =$ Max-Cut is NP-hard

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Question: For general queries, choose $\mathfrak{B}_q$ such that $\mathrm{Aut}(\mathfrak{B}_q)$ is oligomorphic $\Rightarrow$ finitely many orbits of n-tuples for every n.
Can we use some results for finite domains?

Hard resilience problems

pp-construction – a notion of 'expressing' one valued structure in another (generalizes pp-constructions for classical structures)

Fact: If $\mathrm{Aut}(\Gamma)$ is oligomorphic and Γ pp-constructs Δ, then $\mathrm{VCSP}(\Delta)$ reduces to $\mathrm{VCSP}(\Gamma)$ in poly-time.

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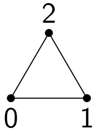

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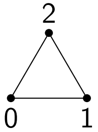

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Corollary

If Aut(Γ) *is oligomorphic and* Γ *pp-constructs* K_3*, then* VCSP(Γ) *is NP-hard.*

Fractional polymorphisms

polymorphism f of $\mathfrak{B}$ – an operation $f : B^n \to B$ that preserves all relations of $\mathfrak{B}$

Idea: CSP($\mathfrak{B}$) is tractable iff it has nice polymorphisms

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Definition (fractional polymorphism)

Γ – valued τ-structure with domain D

A fractional polymorphism of Γ of arity n is a probablity distribution ω on operations $D^n \to D$ such that for every k-ary $R \in \tau$ and $a^1, \ldots, a^n \in D^k$

$$\underbrace{E_\omega[f \mapsto R(f(a^1, \ldots, a^n))]}_{\text{expected value}} \leq \underbrace{\frac{1}{n} \sum_{j=1}^{n} R(a^j)}_{\text{arithmetic mean}} .$$

Tractability conjecture

Known for finite-domain VCSPs:

Theorem

Γ – a finite-domain valued structure

- *If Γ does not pp-construct K_3, then Γ has cyclic fractional polymorphism (essentially Kozik, Ochremiak ('15)).*
- *If Γ has a cyclic fractional polymorphism, then VCSP(Γ) is in P (Kolmogorov, Krokhin, Rolínek ('15)).*

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Theorem (Bodirsky, Lutz, S.)

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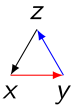
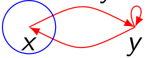
Conjecture: If Γ_q does not pp-construct K_3, then the tractability theorem applies and VCSP(Γ_q) and hence resilience of q is in P.

Examples

- $q_{\text{MC}} := \exists x, y, z\big(R(x,y) \wedge R(y,z)\big)$
- $q_{\text{path}} := \exists x, y, z\big(R(x,y) \wedge S(y,z)\big)$
- $q_{\triangle} := \exists x, y, z\big(R(x,y) \wedge S(y,z) \wedge T(z,x)\big)$
- $q_{\text{new}} := \exists x, y\big(S(x) \wedge R(x,y) \wedge R(y,x) \wedge R(y,y)\big)$





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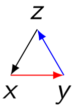

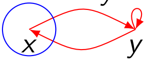

query	**complexity**	**VCSP proof**
q_{MC}	NP-hard (FGIM '20)	Max-Cut/pp-constructs K_3
q_{path}	P (MG)	binary symmetric frac. polymorphism
$q_{\triangle}$	NP-hard (FGIM '15)	pp-constructs K_3
q_{new}	P (BLS)	binary pseudo cyclic frac. polymorphism

References:

- Meliou, Gatterbauer, Moore, Suciu ('10)
- Freire, Gatterbauer, Immerman, Meliou ('15)
- Freire, Gatterbauer, Immerman, Meliou ('20)
- Makhija, Gatterbauer ('22)
- Bodirsky, Lutz, S.

Thank you for your attention

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