Structural Graph Theory, finite bounds and some CSPs

(Finitely bounded expansions of graph classes)

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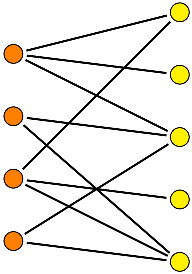
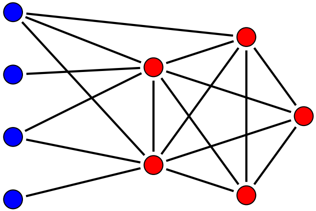
AGK Seminar

Background and motivation

Hereditary classes

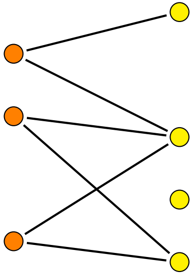


Bipartite graphs



Split graphs

Hereditary classes



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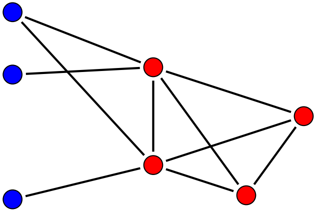
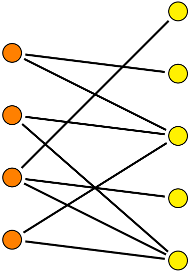
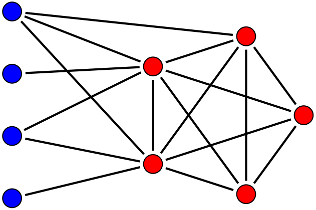


Split graphs

Minimal obstructions



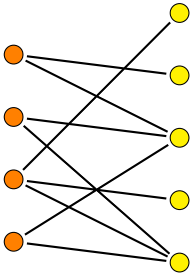
$F_B = \{C_n \colon n \text{ is odd}\}$



$F_S = \{2K_2, C_4, C_5\}$

Minimal obstructions

A **local class** is a hereditary class with a finite set of minimal obstructions



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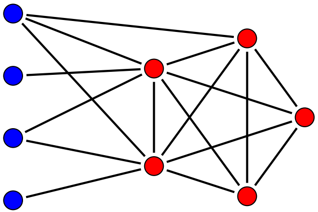

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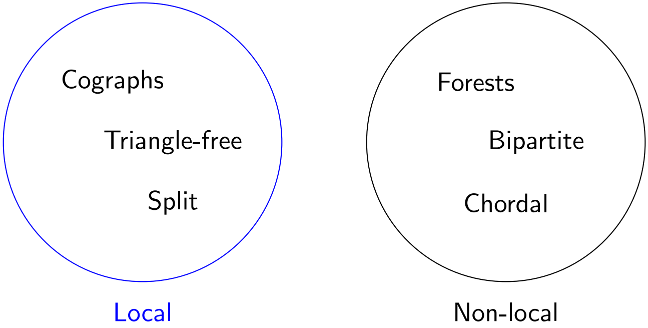

Minimal obstructions

Proposition

For a hereditary class of graphs $\mathcal{C}$, the following statements are equivalent:

- $\mathcal{C}$ is a local class,
- there is a positive integer N such that a graph G belongs to $\mathcal{C}$ if and only if each $H < G$ with $|V(H)| \leq N$ belongs to $\mathcal{C}$, and
- $\mathcal{C}$ is an $\forall_1$-definable class.

Minimal obstructions

Example

The following statements are equivalent:

- G is a triangle-free graph,
- each $H < G$ with $|V(H)| \leq 3$ is triangle-free, and
- $G \models \forall x, y, z(\neg(E(x, y) \wedge E(y, z) \wedge E(y, z)))$

Expressibility by forbidden linear orderings

Corollary (Roy-Gallai-Hasse-Vitaver Theorem, 1962–1968)

A graph G is bipartite if and only if there is a linear ordering of $V(G)$ such that for every vertex v its neighbourhood is contained in $\{y \in V(G) \colon y \geq x\}$ or in $\{y \in V(G) \colon y \leq x\}$.

Exercise

A graph G is a forest if and only if there is a linear ordering of $V(G)$ such that every vertex v has at most one neighbour y such that $x \leq y$.

Theorem (Fulkerson, Gross, 65)

A graph G is a chordal graph if and only if there is a linear ordering of $V(G)$ such that for every vertex v the intersection of $N(v) \cap \{y \in V(G) \colon x \leq y\}$ is a complete graph.

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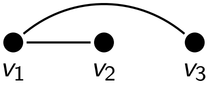

Expressibility by forbidden linear orderings

A class of graphs $\mathcal{C}$ is an **expressible by forbidden linear orderings** if there is a finite set F such that $\mathcal{C}$ is the class of F-free linearly orderable graphs (**FOSG-classes** according to Damaschke)

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Examples (Damaschke, 1990)

The following classes are expressible by forbidden linear orderings

- Chordal graphs (F. & G.)
- Forests
- Split graphs
- k-colourable graphs (RGHV-Theorem)
- Comparability graphs
- Interval graphs

Expressibility by forbidden linear orderings

Theorem (Duffus, Ginn, Rödl, 1995)

For almost all 2-connected linearly ordered graphs $(G, \leq)$, it is NP-complete to decide if a graph G admits an $(G, \leq)$-free linear ordering.

Theorem (Hell, Mohar and Rafiey, 2014)

For any set F of linearly ordered graphs on 3 vertices, it is in P to decide if a graph G admits an F-free linear ordering.

Expressibility by forbidden linear orderings

Feuilloley and Habib, 2020

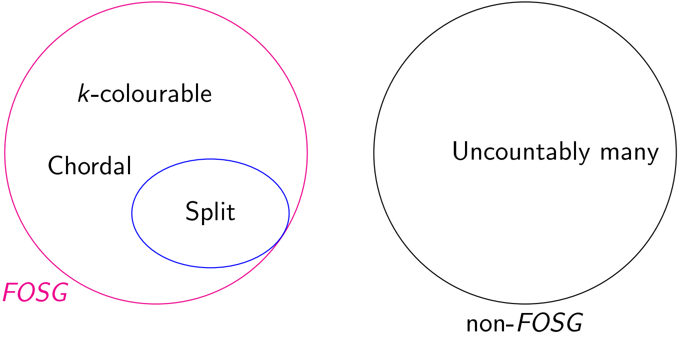
- Characterize all classes of graphs expressible by a set of forbidden linearly ordered graphs on three vertices
- 20 out of these 22 classes can be recognized in linear time
- Forbidden linearly ordered graphs on 4 vertices (2021)

Expressibility by forbidden linear orderings

"Fundamental" problem

Which properties are (not) expressible by linear orderings?

Expressibility by forbidden linear orderings



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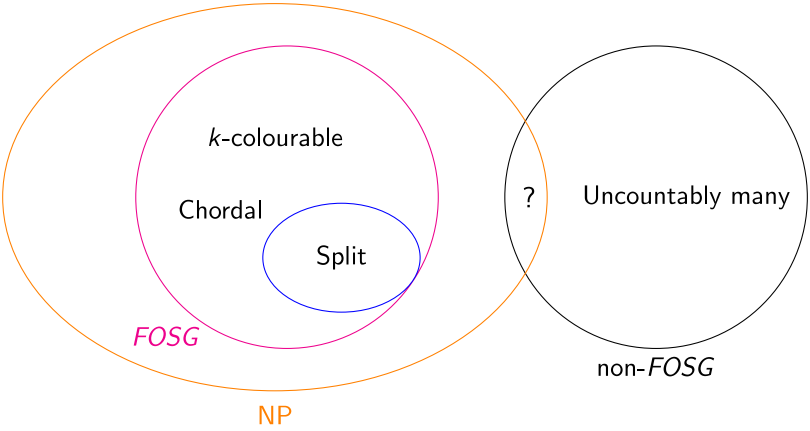

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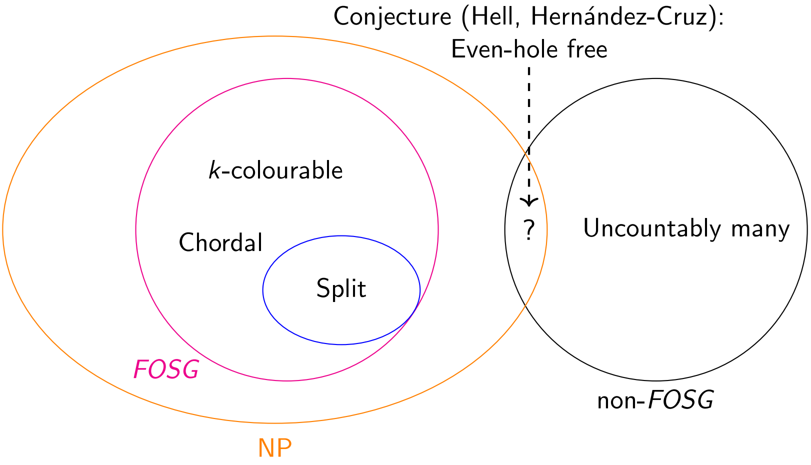

Expressibility by forbidden (acyclic) orientations

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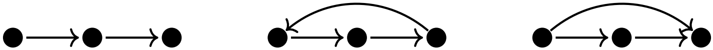
A graph G is a bipartite graph if and only if it admits an F-free (acyclic) orientation



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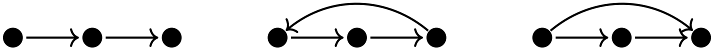

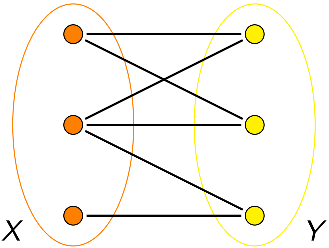

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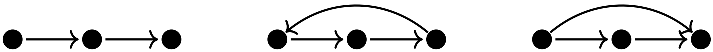

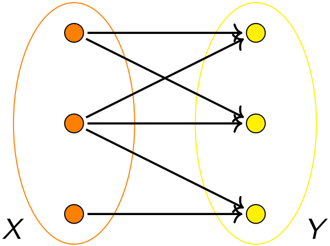

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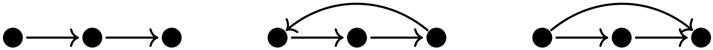

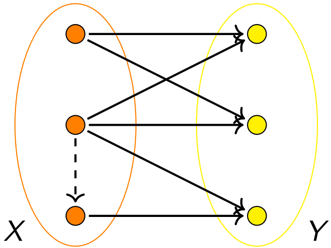

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A class of graphs $\mathcal{C}$ is an **expressible by forbidden (acyclic) orientations** if there is a finite set F such that $\mathcal{C}$ is the class of graphs that admit an F-free (acyclic) orientation — **class of F-graphs (F^*-graphs)** according to Skrien

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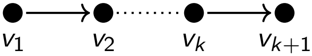
Examples (Skrien, 1980)

- Bipartite graphs (RGHV-Theorem, 1960)
- Trivially perfect graphs
- Proper circular-arc graphs
- Perfectly orientable graphs

Expressibility by forbidden (acyclic) orientations

Roy-Gallai-Hasse-Vitaver Theorem, 1962–1968

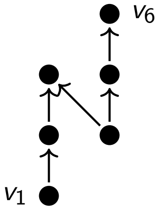
A graph is k-colourable if and only if it admits an orientation with no directed walk on $k+1$ vertices



Expressibility by forbidden (acyclic) orientations

Proposition

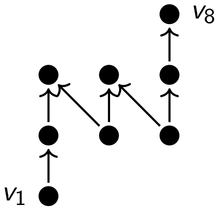
A graph is homomorphic to C_5 if and only if it admits an orientation with no walk,



Expressibility by forbidden (acyclic) orientations

Proposition

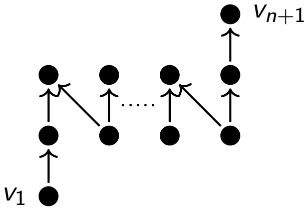
A graph is homomorphic to C_7 if and only if it admits an orientation with no walk:



Expressibility by forbidden (acyclic) orientations

Theorem (G.P., Hernández-Cruz, 21)

A graph is homomorphic to the odd cycle C_n if and only if it admits an orientation with no walk:



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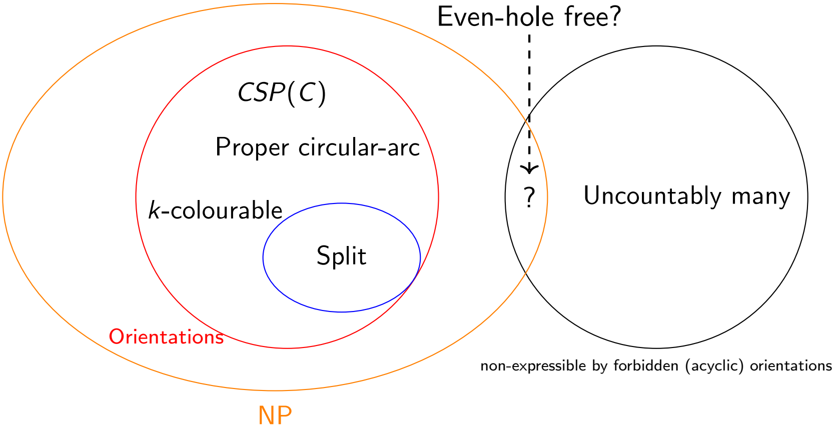

Limitations of expressions by forbidden orientations

Limitations of forbidden orientations

Metaproblem

Which classes are not expressible by forbidden orientations?

Which classes, whose minimal obstructions are cycles, are (not) expressible by forbidden orientations?

Limitations of forbidden orientations

Theorem (G.P., Hernández-Cruz, 22)

Let $\mathcal{P}$ be a property such that its minimal obstructions are cycles. If $\mathcal{P}$ is expressible by forbidden orientations, then the cycles that belong to $\mathcal{P}$ are:

- a finite set and $\{C_k, C_{k+1}, \dots\}$, or
- a finite set and $\{C_{2m}, C_{2(m+1)}, \dots\}$.

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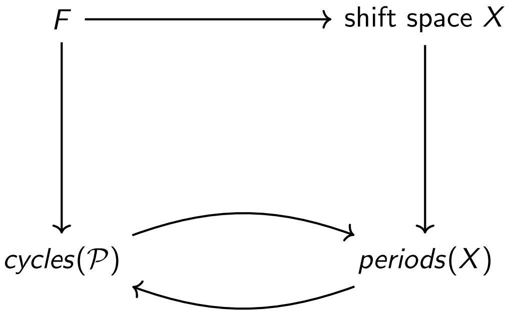
Corollary

The classes of forests, chordal graphs and even-hole free graphs are not expressible by forbidden orientations

Limitations of forbidden orientations

Original proof idea

Let F be a finite set that expresses $\mathcal{P}$ be forbidden orientations



Symbolic Dynamics

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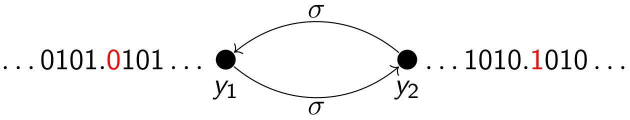
- (X, σ) is called the full-shift
- If $Y \subseteq^* X$, we call (Y, σ) a shift space

Symbolic Dynamics

Let Y be the set of sequences where all consecutive symbols are different, so (Y, σ):



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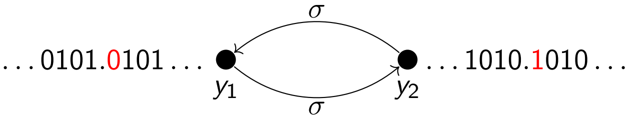

- Y is the set of $\{00, 11\}$-free sequences

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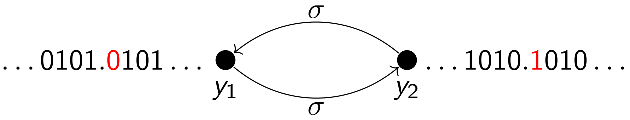

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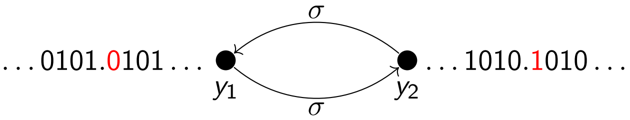
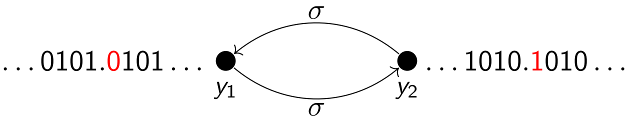


- For every set of words F the collection of F-free sequences is a shift space (X_F, σ),
- (X_F, σ) is a Shift of Finite Type (SFT) when F is finite

Symbolic Dynamics

A point $y \in X_F$ is periodic if $\sigma^n(y) = y$ for some $n > 1$.



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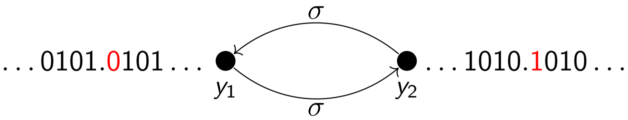

- If n is a period of X_F, then rn is a period of X_F.
- The set of periodic points of a SFT is dense.

Shift spaces and forbidden orientations

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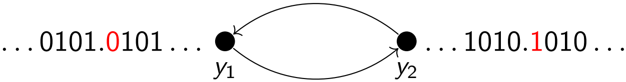

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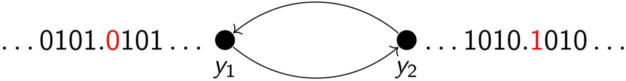
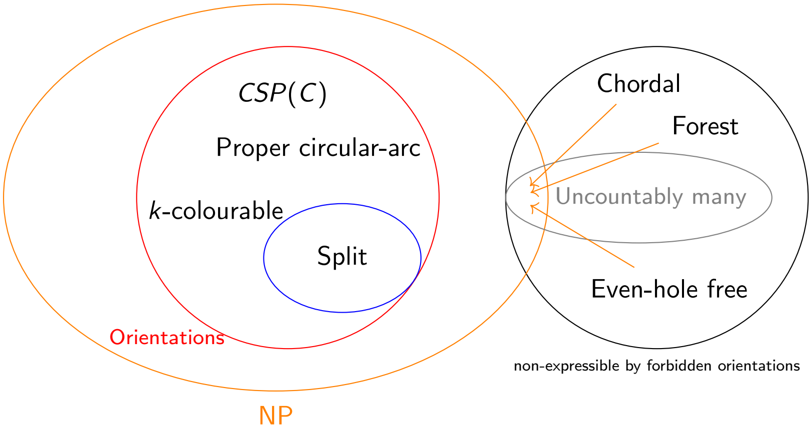


- For every $n \geq 4$, the cycle $C_n \in \mathcal{B}$ if and only if $n \in per(X_F)$

Shift spaces and forbidden orientations



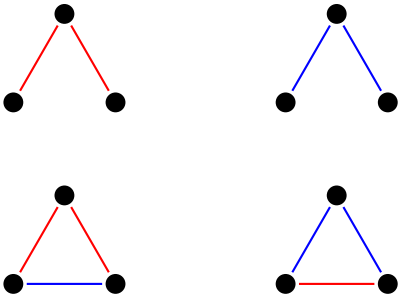
Other characterizations by “equipped” graphs?

Example 1

A graph G is a complete multipartite graph if and only if there is a T_0-topology τ in $V(G)$ such that $xy \in E(G)$ if and only if x and y are separable in $(V(G), \tau)$.

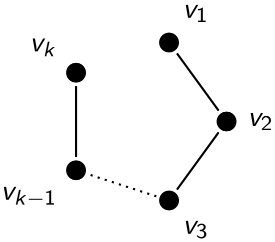
Example 2 (B., G.P., H.C., J., 23+)

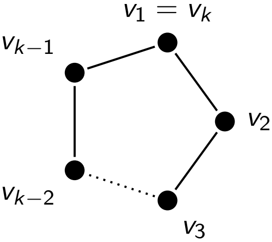
A graph G is the line graph of a bipartite graph if and only if there is a 2-edge colouring of G that avoids



Example 3 (G.P., Hell, Hernández-Cruz, 22)

A graph G admits a circular ordering with no circular walk on $k+1$ vertices if and only if $\chi_c(G) < k$





Possible unified framework?

Local expressions of hereditary classes

Local expressions

Ingredients from Model Theory

- Finite relational signatures L
- Hereditary classes of L-structures
- Relative local classes of L-structures
- Quantifier-free formulas
- Quantifier-free lattice $(L, \wedge, \vee, \neg)$

Local expressions

A **concrete functor** $F\colon \mathcal{C} \to \mathcal{D}$ (between categories of relational structures) is a functor such that $F(\varphi) = \varphi$ for each embedding φ

$$\begin{array}{ccc} A & \xrightarrow{F} & F(A) \\ {\scriptstyle\varphi}\big\uparrow & & {\scriptstyle\varphi}\big\uparrow \\ B & \xrightarrow{F} & F(B) \end{array}$$

Forgetful functors are concrete functors

$$\begin{array}{ccc} (G, \leq) & \longrightarrow & G \\ {\scriptstyle\varphi}\big\uparrow & & {\scriptstyle\varphi}\big\uparrow \\ (H, \leq) & \longrightarrow & H \end{array}$$

Local expressions

Given a surjective functor $F\colon \mathcal{C} \to \mathcal{D}$ we say that a class $\mathcal{D}' \subseteq \mathcal{D}$ is **locally expressible** by F if there is a local class $\mathcal{C}' \subseteq \mathcal{C}$ such that $F[\mathcal{C}'] = \mathcal{D}'$

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} \!\!\!\!\rightarrow & \mathcal{D} \\ \uparrow & & \uparrow \\ \mathcal{C}' & \xrightarrow{F} & \mathcal{D}' \end{array}$$

Example

A class is locally expressible by sh_L if and only if it is expressible by forbidden linear orderings

$$\begin{array}{ccc} \mathcal{LOR} & \xrightarrow{sh_L} \!\!\!\!\rightarrow & \mathcal{G} \\ \uparrow & & \uparrow \\ \mathcal{B}' & \xrightarrow{sh_L} \!\!\!\!\rightarrow & \textit{Bipartite} \end{array}$$

Local expressions

Given a surjective functor $F\colon \mathcal{C} \to \mathcal{D}$ we say that a class $\mathcal{D}' \subseteq \mathcal{D}$ is **locally expressible** by F if there is a local class $\mathcal{C}' \subseteq \mathcal{C}$ such that $F[\mathcal{C}'] = \mathcal{D}'$

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} \!\!\!\!\rightarrow & \mathcal{D} \\ \uparrow & & \uparrow \\ \mathcal{C}' & \xrightarrow{F} & \mathcal{D}' \end{array}$$

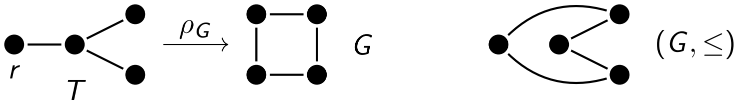
Examples

A class is locally expressible by S if and only if it is expressible by forbidden orientations

$$\begin{array}{ccc} \mathcal{OR} & \xrightarrow{S_{|\mathcal{OR}}} \!\!\!\!\rightarrow & \mathcal{G} \\ \uparrow & & \uparrow \\ CSP(AC) & \xrightarrow{S_{|\mathcal{OR}}} \!\!\!\!\rightarrow & CSP(C) \end{array}$$

Local expressions

Tree-layouts (Paul, Protopapas, 23+) & Genealogical graphs



A **tree-layout** consists of a rooted tree together with a bijection such that YYY

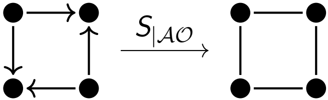
- generalize linear orderings
- induced substructure defined by ancestor relation

A **genealogical graph** is a partially ordered graph such that YYY

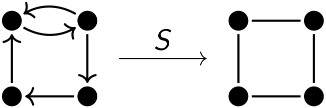
- equivalent to tree-layouts
- induced relational structure

Local expressions

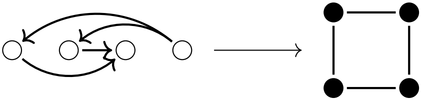
More examples



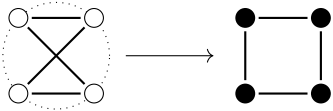
Acyclic oriented graphs



Digraphs



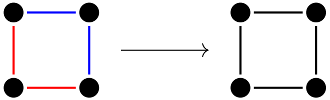
Linearly ordered oriented graphs



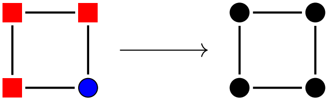
Circularly ordered graphs

Local expressions

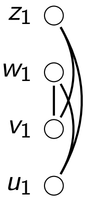
More examples

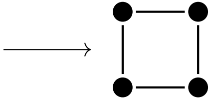


2-edge-coloured graphs

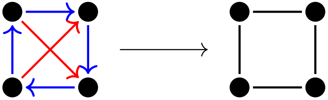


Vertex-coloured graphs





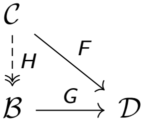
Partially ordered graphs



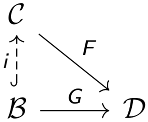
2-arc-coloured tournaments

Local expressions

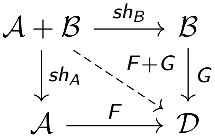
Algebraic constructions and relations



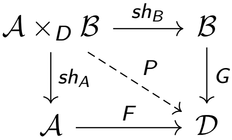
If G factors F, then $ex(G) \subseteq ex(F)$



If F extends G, then $ex(G) \subseteq ex(F)$

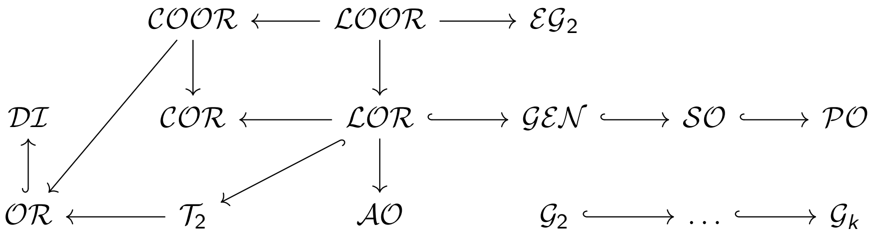


$ex(F) \cup ex(G) \subseteq ex(F+G)$



$\mathcal{P} \in ex(F), \mathcal{Q} \in ex(G) \to \mathcal{P} \cap \mathcal{Q} \in ex(P)$

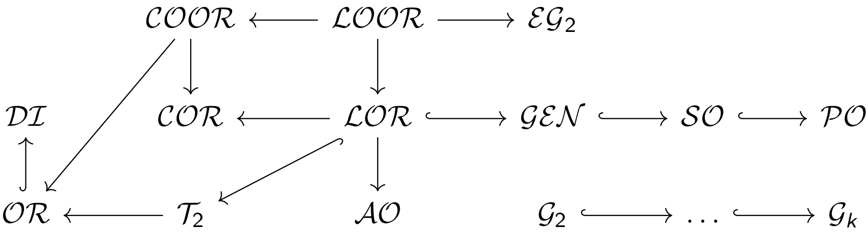
Local expressions



Which local expressions are worth studying?

- By tradition → Linear orderings & orientations
- By authority (if Pavol studies it, it is worth it) → Linear & circular orderings
- By expressive power → ???

Local expressions



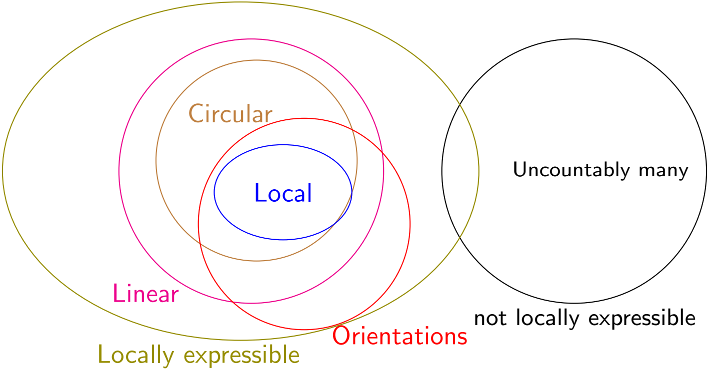
Truth is . . .

All of them are!

Local expressions

A class $\mathcal{D}$ is a **locally expressible** if it is locally expressible by some functor F. Equivalently, $\mathcal{D}$ is a locally expressible class if there is a local class $\mathcal{C}$ and a surjective functor $F\colon \mathcal{C} \to \mathcal{D}$.

Local expressions



- There are countably many locally expressible classes
- Locally expressible classes are closed under unions and intersections!
- *Certificates for locally expressible classes?*

Local expressions

Theorem (G.P., 23+)

For every concrete functor $F\colon \mathcal{C} \to \mathcal{D}$, there is an algorithm that constructs $F(X)$ in polynomial-time (with respect to $|V(X)|$) for each $X \in \mathcal{C}$.

Corollary

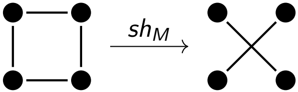
If $\mathcal{C}$ is a locally expressible class, then $\mathcal{C}$ is in NP.

Local expressions

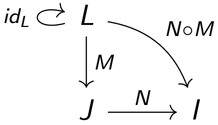
Given a pair of relational signatures L and J, and **emulation** of L in J is lattice homomorphism $M\colon (L, \wedge, \vee, \neg) \to (J, \wedge, \vee, \neg)$ such that for every $R \in L$ of arity m, the formula $M(R)$ has exactly m free variables. Every emulation defines a polynomial time computable functor $sh_M\colon Mod_J \to Mod_L$.

Example

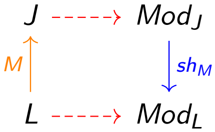
Consider the emulation $M\colon (\{E\}, \wedge, \vee, \neg) \to (\{E\}, \wedge, \vee, \neg)$ defined by $M(E) = \neg E(x, y)$.



Local expressions



Emulations define a category

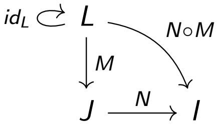


Contravariant functor to category of categories relational structures

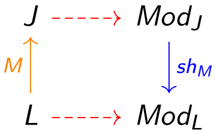
Properties of $\dashrightarrow$

- M is injective if and only if sh_M is surjective
- sh_M is a polynomial-time computable functor
- Faithful contravariant functor bijective on objects

Local expressions



Emulations define a category



Contravariant functor to category of categories relational structures

Theorem (G.P., 23+)

Consider a pair of relational signature L and J. If $F\colon Mod_J \to Mod_L$ is a concrete functor, then there is an emulation $M\colon L \to J$ such that $F = sh_M$.

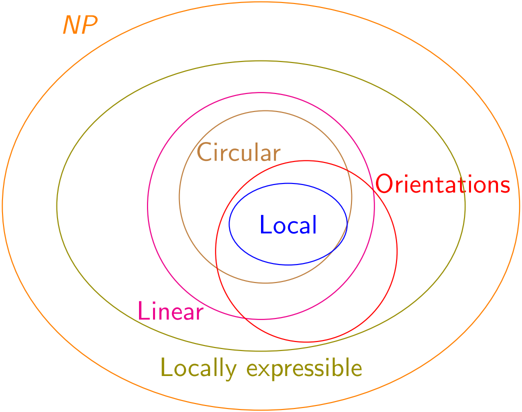
Corollary 1

The categories **Em** and $\mathbf{Mod}^{opp}$ are isomorphic categories.

Local expressions

Corollary 2

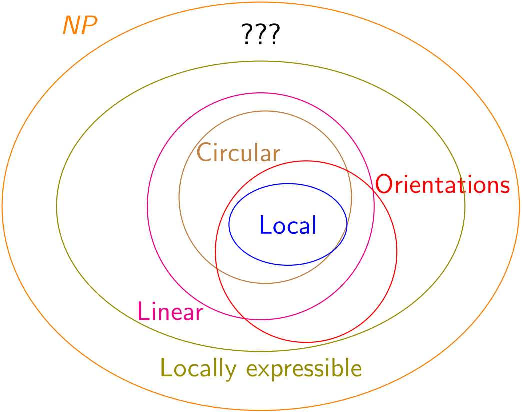
Local expressions yield polynomial-time certificates for membership problem.



Local expressions

Fundamental problem

What can be certified by local expressions?



Future research

Three generic problems for a functor $F\colon \mathcal{C} \to \mathcal{G}$

Characterization Problem. Consider a finite set $X \subseteq \mathcal{C}$. Characterize the class of graphs G for which there is an X-free structure $A \in \mathcal{C}$ such that $F(A) = G$.

Complexity Problem. Consider a finite set $X \subseteq \mathcal{C}$. Determine the complexity of deciding if for some input graph G there is an X-free structure $A \in \mathcal{C}$ such that $F(A) = G$.

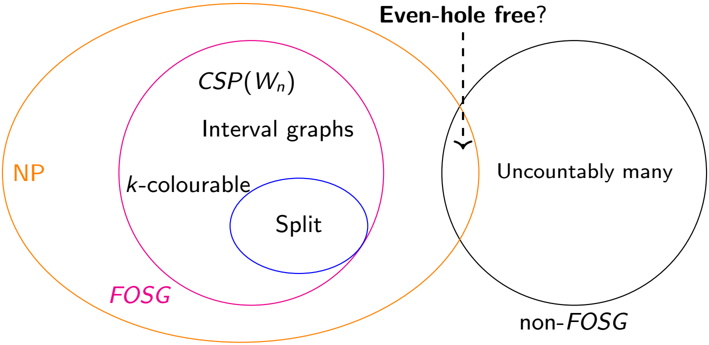
Expressibility Problem. Consider a class of graphs $\mathcal{P}$. Determine if $\mathcal{P}$ is locally expressible by F.

Characterization & Complexity Problems — Oriented graphs

- Find all minimal obstructions of perfectly orientable graphs (Skrien, 82)
- Characterize transitive perfectly orientable
- Complexity of recognizing transitive perfectly orientable

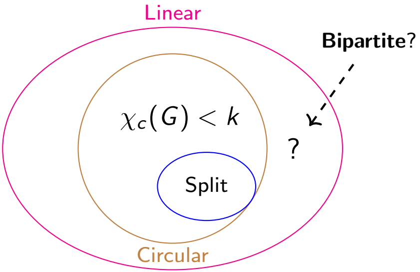
Expressibility Problem — Linear orderings

- Exhibit a *well-known* class of graphs that is not an *FOSG*-class (Damaschke, 90)
- Is there a class expressible by forbidden orientations but not by forbidden orderings?
- Is the class of **even-hole free** graphs an *FOSG*-class? (Hell, Hernández-Cruz, 16)



Expressibility Problems — Circular orderings

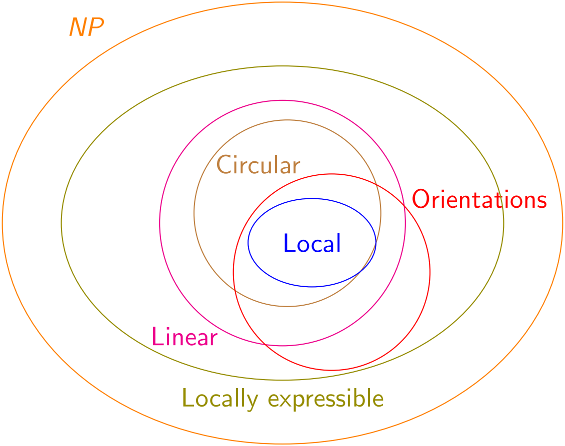
- Do forbidden linear orderings have a larger expressive power than forbidden circular orderings?
- Is the class of **bipartite graphs** expressible by forbidden circular orderings?



"Metaquestion"

- What can be certified by local expressions?
- Find an example of a hereditary (graph) class $\mathcal{C} \in NP$ that is not a locally expressible class — equivalently, $\mathcal{C} \notin SNP$.

Thank you!



NP
Circular
Orientations
Local
Linear
Locally expressible