Model-theoretic Challenges in Constraint Satisfaction

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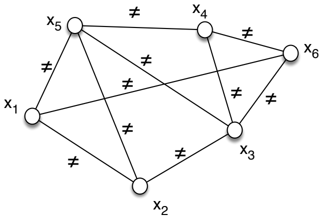

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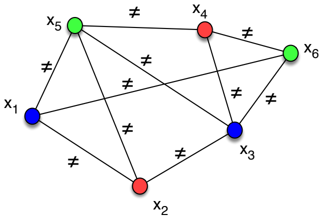

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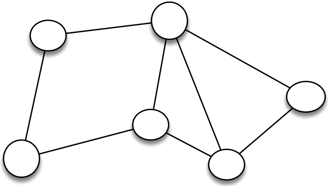

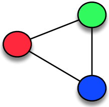

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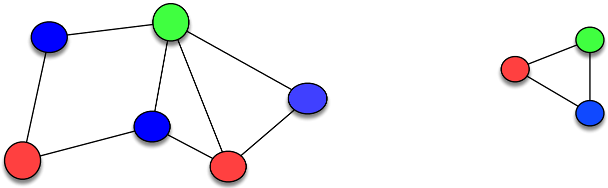

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Theorem. B.+Grohe'08: Every decision problem is equivalent to a CSP (under polynomial-time Turing reductions)

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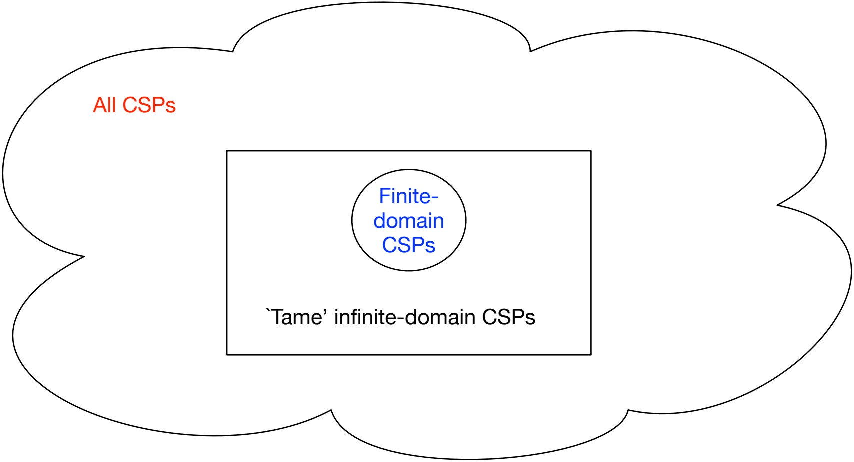

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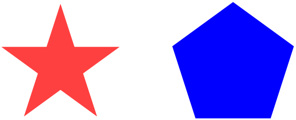

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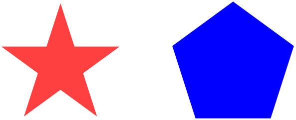

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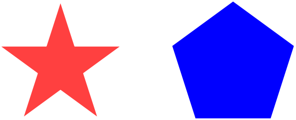

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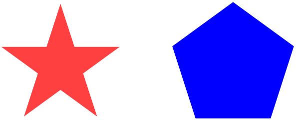

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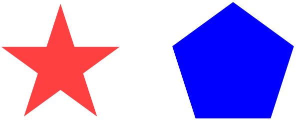

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- $\mathrm{Pol}(\mathfrak{B})$ is a clone: contains projections and closed under composition.

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Theorem (Bulatov'17, Zhuk'17). If $\text{Pol}(\mathfrak{C}, c_1, \dots, c_n)$ does does not have a homomorphism to $\text{CSP}(K_3)$, then $\text{CSP}(\mathfrak{B})$ is in P.

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Equivalently:
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(2) Reconstruction Conjecture. Let $\mathfrak{A}$ and $\mathfrak{B}$ be reducts of structures that are homogeneous with finite relational signature. Then

- $\text{Pol}(\mathfrak{A}) \simeq \text{Pol}(\mathfrak{B}) \Rightarrow \text{Pol}(\mathfrak{A}) \simeq_t \text{Pol}(\mathfrak{B})$
- $\text{Aut}(\mathfrak{A}) \simeq \text{Aut}(\mathfrak{B}) \Rightarrow \text{Aut}(\mathfrak{A}) \simeq_t \text{Aut}(\mathfrak{B})$.

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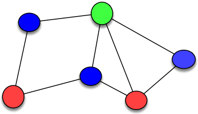

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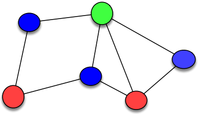

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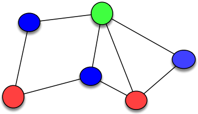

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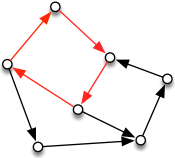

Countably Categorical Structures for MSO sentences

Theorem (B.+Knäuer+Rudolph'21).

Let $\mathfrak{B}$ be such that CSP$(\mathfrak{B})$ is in MSO. Then there exists an ω-categorical structure $\mathfrak{C}$ such that CSP$(\mathfrak{B}) =$ CSP$(\mathfrak{C})$.

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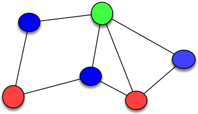

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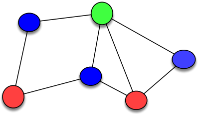

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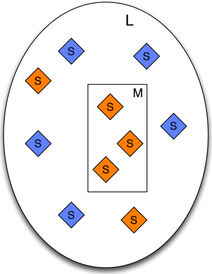

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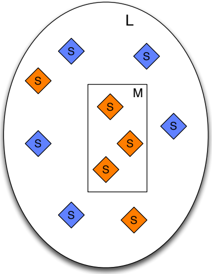

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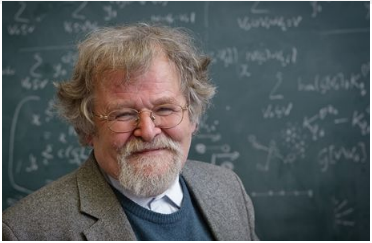

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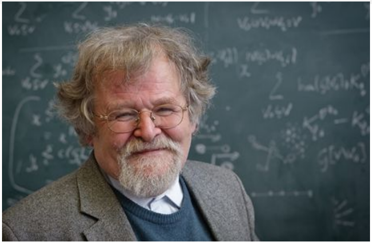

Example. $(\mathbb{Q}; <)$ is Ramsey.

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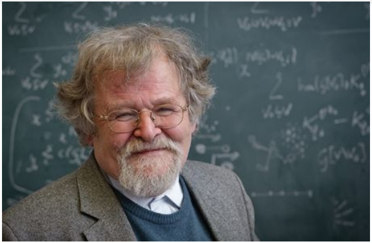

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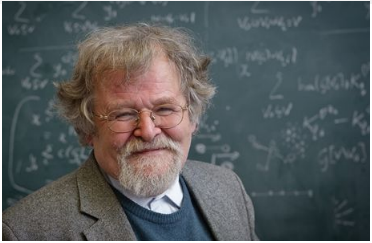
Reformulation of Ramsey's theorem!

The Ramsey Property



Definition [Nešetřil].
A structure $\mathfrak{B}$ is Ramsey if
$\mathfrak{B} \to (\mathfrak{M})^{\mathfrak{S}}_c$ for all finite $\mathfrak{S}, \mathfrak{M} \hookrightarrow \mathfrak{B}$
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Theorem (Kechris, Pestov, Todorcevic'05).

A homogeneous structure $\mathfrak{B}$ is Ramsey if and only if $\mathrm{Aut}(\mathfrak{B})$ is extremely amenable, i.e., every continuous action on a compact Hausdorff space has a fixed point.

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(3) Ramsey Expansion Conjecture. Every homogeneous structure with finite relational signature has a finite homogeneous Ramsey expansion.

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- Additionally assume that $\mathfrak{C}$ is NIP and has binary signature.